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ABSTRACT

This is volume one of a three-volume SMSG junior high school mathematics text. This volume contains units concerned with the structure of the number systems of arithmetic. Unit topics include: (1) numeration; (2) natural numbers and zero; (3) factoring and primes; (4) supplementary tests for divisibility and repeating decimals; (5) the non-negative rational numbers; and (6) mathematical systems. (MP)

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JUNIOR HIGH SCHOOL MATHEMATICS UNITS

VOLUME I, NUMBER SYSTEMS

Prepared by the SCHOOL MATHEMATICS STUDY GROUP
Under a grant from the NATIONAL SCIENCE FOUNDATION

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1959

These units were prepared in the summer of 1958 at Yale University by a School Mathematics Study Group writing team. The units were taught in a number of classes during the academic year 1958-59. No major editing of them has been attempted, but typographical and other errors have been corrected.

This volume includes the units concerned with the structure of the number systems of arithmetic. These are units II, III, IV, IVa, V, XIV in the numbering system originally used.

TABLE OF CONTENTS

		Page
Unit II	Numeration.	1
Unit III	Natural Numbers and Zero.	23
Unit IV	Factoring and Primes.	55
Unit IVa	Supplementary Tests for Divisibility and Repeating Decimals. . .	75
Unit V	The Non-negative Rational Numbers.	90
Unit XIV	Mathematical Systems.	118

UNIT II

NUMERATION

HOW OTHER PEOPLES WROTE NUMERALS

What does the numeral 358 mean?

The 3 means 3 hundreds, the 5 means 5 tens, and the 8 means 8 ones. 358 can also be written as 3 hundreds + 5 tens + 8 ones. Adding these parts, the whole numeral means three hundred fifty eight.

People did not always use this system for writing numerals. When primitive people kept a record of a number, they often did it by making scratches in the dirt or on a stone, cutting notches in a stick, or tying knots in a rope. They kept track of their sheep or other animals by placing pebbles in a pile, one for each animal. Later they knew that sheep were missing if there was not one sheep for each pebble. We do much the same thing when we count the votes in a class election, making one mark for each vote, like this: 𝍸 ||| . The important thing about these ways of recording numbers is that one mark stands for one object, and there are as many marks as there are objects. They did not have any marks which stood for several things, such as our "8" to stand for "||||||||" or the Roman "V" to stand for "|||||."

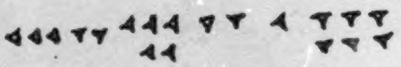
Much later, people began to use a single symbol to stand for several objects. The Egyptians used symbols of this kind. They had different symbols for 1, 10, 100, 1000, 10,000, 100,000 and 1,000,000.

Their symbols are shown below.

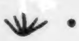
Number	Symbol	Resembles
1	𓏺	Stroke
10	𓎆	Heel bone
100	𓍢	Coiled rope
1000	𓆼	Lotus flower
10,000	𓂭	Bent reed
100,000	𓆐	Burbot fish
1,000,000	𓁨	Astonished man

They repeated the symbol to show other numbers. Their symbol for 100 was "𓍢", resembling a coiled rope, and they wrote seven hundred as "𓍢𓍢𓍢𓍢𓍢𓍢𓍢". The order in which the symbols were written was not important, and hundreds could be written either before or after thousands. They could write "35" either as 𓎆𓎆𓎆𓏺𓏺𓏺𓏺𓏺 or 𓏺𓏺𓏺𓏺𓏺𓎆𓎆𓎆. "2341" was written 𓆼𓆼𓍢𓍢𓍢𓎆𓎆𓎆𓎆𓏺 or 𓍢𓍢𓍢𓆼𓆼𓎆𓎆𓎆𓎆𓏺 or 𓏺𓎆𓎆𓎆𓎆𓍢𓍢𓍢𓆼𓆼. Notice that the numerical value of an Egyptian numeral is the sum of the numbers represented by the individual numerals.

The Babylonians, who lived long ago in the part of Asia which we call the Middle East, had an interesting way of writing numerals. They did their writing on clay tablets with a wedge-like instrument called a stylus. Their symbol for 1 was 𒁹 and their symbol for 10 was 𒌋. They repeated and combined these symbols to write the numbers up to 59. For example, "𒌋𒁹𒁹" meant twelve, and "𒌋𒌋𒌋𒌋𒁹𒁹𒁹𒁹𒁹" meant forty-five. To write numbers larger than 59 they used the same symbols, but the position in which they were written changed their value. They used the number sixty in the same way that we use ten. If we write 452, it means (4 x 10 x 10) + (5 x 10) + (2 x 1). When the Babylonians wrote

 it meant (32 x 60 x 60) + (52 x 60) + (16 x 1). Since they had no symbol for zero until about 200 B. C., they sometimes left empty spaces in their numerals where we would write zeros. It is thought that our way of dividing an hour into sixty minutes and a minute into sixty seconds is related to the way the Babylonians used sixty in writing large numbers.

1. How would the Babylonian numeral above be written in our numerals?
2. Since order made no difference in the meaning of the Egyptian numerals, in what other ways than those shown might they have written 2341? How might they have written 37? 1,111,111?

You have often seen Roman numerals. This system used strokes, which many historians believe were pictures of fingers, for the numbers from one to four, like this: | || ||| |||| . Then they used a hand, for five, like this:  . Gradually they began to leave out some of these marks and wrote five this way: V. They put two hands together for ten, X , which later became X. To write other numbers they combined these by addition. For example, XVII = 10 + 5 + 1 + 1 = 17; CLXIIII = 100 + 50 + 10 + 1 + 1 + 1 + 1 = 164

3. Write the following in Roman numerals:

   (a) 51 (b) 75 (c) 160 (d) 512 (e) 1958

4. Write the following in ordinary Hindu-Arabic numerals:

   (a) VIII (b) CV (c) LXI (d) MCCL (e) DIII

Much later, the Romans began to use subtraction to write some numbers, and wrote four as IV, nine as IX, and forty as XL. As you know, L means fifty, C means 100, D means 500 and M means 1000. Sometimes they wrote a bar over X, C, and M, and that multiplied the value by

1000. $\overline{X}$ means 10,000, $\overline{C}$ means 100,000, and $\overline{M}$ means 1,000,000.

5. Look up the Greek, Hebrew, and Chinese numerals. Make charts of these notations for display in your classroom.
6. How would the sum of 3 and 5 have been written by a Greek child? How would he write 30 + 50? How does this compare with the usual way of solving these two problems?
7. Look up Al-Khwarizmi, Gerbert (Pope Sylvester II), Adelhard of Bath, and Leonardo of Pisa (Fibonacci).
8. Find out where our word "digit" comes from.
9. Choose a number and write it in as many different kinds of numerals as you can.

OUR DECIMAL NUMERALS

Our way of writing numerals was invented in India, and brought to Europe by Arabs. For this reason they are called Hindu-Arabic numerals, although the symbols the Arabs use now are different from our symbols. The important characteristics about this system of writing numerals are:

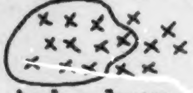
(1) each symbol is the name of a number;

(2) the position of a symbol in a numeral tells the size of a group; and

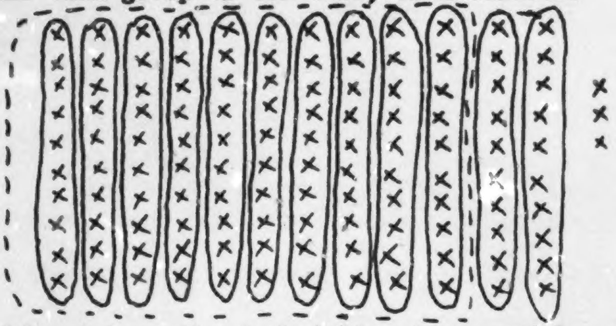
(3) there is a symbol for zero, which is used to fill places which would otherwise be empty and might lead to misunderstanding.

For example, 3 is the numeral for a set of three things. 30 is the numeral for 3 collections of ten things each, or thirty things. 300 is the numeral for 3 collections of one hundred things each, or three hundred things. Addition, multiplication, subtraction, and division are usually easier to perform with our system than with other systems, as you will see if you try to add or multiply using Egyptian, Roman, or Babylonian numerals.

The system we use for writing numerals is called a decimal system. The word "decimal" comes from a Latin word which means ten. We can write the numeral for any number, however large or small, by using just ten number symbols, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. This is possible, because the way we write numerals makes use of place value; that is, the position in which a symbol is written, as well as the symbol, determines the number for which it stands. The 3 in 358 stands for three hundreds, the 5 for 5 tens or fifty, and the 8 for eight units. When we count a group of objects, we usually group them in tens. For example, in the x's below a line is drawn around a group of ten x's. So we say there are 16 x's -- one group of ten, and six more.



Suppose we had a large group of x's. We can draw lines around groups of ten until fewer than ten are left. Then we can draw another line around each group of ten tens, or a hundred.



In this picture the dashed line is around 1 group of 10 tens or 1 hundred; there are also 2 tens, and 3 units. So there are (1x10x10) + (2x10) + (3x1), or 123.

10. Draw twenty-seven x's, and draw lines around groups of ten to show what 27 means.

11. Draw a group of x's in which all of the x's are in groups of 10. Write the numeral for this group.

12. Write the meaning of these numerals in the way shown for 48.

48 = (4x10) + (8x1)

(a) 354

(b) 6421

(c) 709

(d) 320

(e) 2000

13. Copy and complete the following multiplication table, using Roman numerals.

	I	V	X	L	C	D	M
I							
V							
X							
L							
C							
D							
M							

When we add numbers we make use of groups of ten also. When we say "How much is 9 + 7?", we mean "How many tens and ones?" So we do not say, "9+7=5+11," although that would be correct. We can regroup as follows:

9+7 =		9+7 =
9+(1+6) =	or	(6+3)+7 =
(9+1)+6 =		6+(3+7) =
10+6 =		6+10 =
16		16

We say our system of writing numerals has the base ten.

Starting at ones' place, each place to the left is given a value 10 times as large as the place before. The places from right to left have the values shown below.

10x10x10x10x10	10x10x10x10	10x10x10	10x10	10	1

Often we write these values more briefly by using a small numeral to the right and above the 10. This numeral shows how many 10's are multiplied together. In this way, the values of the places are written:

10^5	10^4	10^3	10^2	10^1	1

and are read

10 to the fifth power, 10 to the fourth power, 10 to the third power (or 10 cubed), 10 to the second power (or 10 squared), 10 to the first power, 1.

Numerals used as the 5, 4, 3, 2, and 1 are used above are called "exponents" and the numeral with which they are used (in this case, 10) is called the base. The number represented by the entire expression, such as 10^4, is called a power. 4^3 is a short way to write 4x4x4.

14. What is a short way to write 3x3x3x3?

15. What is the meaning of 5^4?

16. What number is represented by 4^3?

17. Which represents the larger number: 4^3 or 3^4?

The meaning of 352 may now be written using exponents, like this: $(3\text{x}10^2) + (5\text{x}10^1) + (2\text{x}1)$.

18. Write the numbers below using exponents.

(a) 468 (b) 5324 (c) 7062 (d) 59,120

Probably the reason that we use a numeral system with ten as base is that people have ten fingers, and when primitive men began to count their possessions they counted on their fingers. This accounts for the fact that the ten symbols, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 are called "digits." We speak of these symbols as digits when we wish to refer to them apart from the longer numerals in which they are used. For example, the "digits" in 458 are 4, 5, and 8. The Celts, who lived in Europe more than 2,000 years ago, used twenty as base, and so did the Mayans in Central America. Can you think of a reason? What special word do we sometimes use for twenty? If Martians had a different number of fingers they might use some other number as the base of their numerals. Let us see how systems with bases other than ten work.

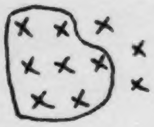
19. Look up the French words for "eighty" and "ninety." Do you know an English word which indicates that some people used to count by twenties? (Think of Lincoln's Gettysburg Address.)

NUMERALS IN BASE SEVEN

Suppose we have a system with seven as base. What symbols shall we have?

0, 1, 2, 3, 4, 5, 6.

There are seven digits or symbols needed for a system of numerals to the base seven just as there were ten needed for base ten. Let us group the x's below so as to write the number of x's in base seven.



We draw a line around seven x's, and see that there is 1 group of seven, and 3 more. There are 13_{seven}. We write the "seven" to show what base we are using. "13_{seven}" means 1 group of seven and 3 ones. We read 13_{seven} as one-three, not as thirteen. Up to 66_{seven}, you may think of the numbers in terms of weeks and days. When no base is written, we understand the numeral is written in base ten.

20. Draw x's and group them with lines to show the meaning of

    (a) 25_{seven} (b) 32_{seven} (c) 40_{seven} (d) 123_{seven}

    Then write each of the above in decimal notation.

21. Numerals in base seven can be written out like this:

    246_{seven} = (2 x seven^2) + (4 x seven^1) + (6 x one) =

    (2 x seven x seven) + (4 x seven) + (6 x one) = ?

    How would you write this numeral in the usual decimal notation?

    Write these numerals with exponents in the above way:

    (a) 56_{seven} (b) 241_{seven} (c) 500_{seven} (d) 4120_{seven}

    Then express each in decimal notation.

22. Below is the beginning of a chart of the numerals for the

numbers from 0 to 110 (base ten). From here on, in indicating base ten, write it in words ("base ten") rather than as a numeral. This is necessary since 10 means one of the base, which may be 7 or any number. Finish the chart, as you will need it later.

0	10	20	30	40	50	60	70	80	90	100	110
1	11	21	31	41	51	61	71				
2	12	22	32	42	52	62					
3	13	23	33								
4	14	24									
5	15										
6	16										
7	17										
8	18										
9	19										

23. Copy and complete the following chart of base seven numerals, showing the numerals from 0_{seven} to 110_{seven}.

0	10	20	30	40	50	60
1	11	21				
2	12			42		
3	13					
4	14					
5	15				55	
6	16					

Note that 10_{seven} is read one-zero, not as ten and that 11_{seven} is read one-one, not as eleven, etc.

Using Base Seven Numerals

24. When we learn to add numbers expressed in base ten we learn 100 "basic" combinations. These combinations are arranged in the chart below. Finish the chart.

		Second addend									
		0	1	2	3	4	5	6	7	8	9
First addend	0	0	1	2	3	4	5	6	7	8	9
	1		2	3	4	5	6	7	8	9	10
	2			4	5	6	7	8	9	10	11
	3				6	7	8	9	10	11	12
	4					8	9	10	11	12	13
	5						10	11	12	13	14
	6							12	13	14	15
	7								14	15	16
	8									16	17
	9										18

25. Compare the part of the table you have filled in with the part which is there now. What do you notice about the two parts?

26. Finish the chart below, showing the sums in base seven.

		Second addend						
		0	1	2	3	4	5	6
First addend	0							
	1							
	2							
	3							
	4							
	5							
	6							

27. When you add numbers expressed in base seven, how many basic combinations are there?

28. Draw lines in the addition chart for base seven like the lines in the chart for base ten. Are the two parts of the table alike?

29. Add these numbers in base ten:

$$\begin{array}{r} 25 \\ \underline{48} \end{array}$$

As you know, you "carry" when the sum of a column is ten or more.

$$\begin{array}{rcl} 25 & = & 2 \text{ tens} + 5 \text{ ones} \\ \underline{48} & = & \underline{4 \text{ tens} + 8 \text{ ones}} \\ & & 6 \text{ tens} + 13 \text{ ones} = 7 \text{ tens} + 3 \text{ ones} \end{array}$$

Add:

$$\begin{array}{rcl} 24_{seven} & = & 2 \text{ sevens} + 4 \text{ ones} \\ \underline{35}_{seven} & = & \underline{3 \text{ sevens} + 5 \text{ ones}} \\ & & 5 \text{ sevens} + \text{nine ones} = 6 \text{ sevens} + 2 \text{ ones} = 62_{seven} \end{array}$$

30. Is it possible to "carry" in addition in base seven just as you do in addition in base ten?

31. In base ten, you "carry" when the sum of a column is ? or more.

32. In base seven, you "carry" when the sum of a column is ? or more.

33. Add these numbers in base seven.

(a) $\begin{array}{r} 42_{seven} \\ \underline{13}_{seven} \end{array}$ (b) $\begin{array}{r} 65_{seven} \\ \underline{11}_{seven} \end{array}$ (c) $\begin{array}{r} 32_{seven} \\ \underline{25}_{seven} \end{array}$ (d) $\begin{array}{r} 254_{seven} \\ \underline{105}_{seven} \end{array}$ (e) $\begin{array}{r} 435_{seven} \\ \underline{625}_{seven} \end{array}$

34. For Ex. 33c, write out the addition to show how your answer was obtained. Express the numbers in Ex. 33 to base ten, and check your answers by adding in the usual way.

35. Subtract: $\begin{array}{r} 43_{seven} \\ \underline{15}_{seven} \end{array}$ Was your remainder 25_{seven}?

To subtract $\begin{array}{r} 43_{seven} \\ \underline{15}_{seven} \end{array}$, think "4 sevens + 3 ones = 3 sevens + ten ones

$$\begin{array}{rcl} & & \\ 1 \text{ seven} + 5 \text{ ones} & = & \underline{1 \text{ seven} \;+\; 5 \text{ ones}} \\ & & 2 \text{ sevens} + 5 \text{ ones} \end{array}$$

36. Subtract these numbers in base seven:

(a) $\begin{array}{r} 56_{seven} \\ \underline{14}_{seven} \end{array}$ (b) $\begin{array}{r} 61_{seven} \\ \underline{35}_{seven} \end{array}$ (c) $\begin{array}{r} 34_{seven} \\ \underline{25}_{seven} \end{array}$ (d) $\begin{array}{r} 456_{seven} \\ \underline{263}_{seven} \end{array}$

37. For Ex. 36c, show how the 34_{seven} is changed, in order to make it easier to subtract.

38. Complete the multiplication chart below for numbers in base seven. Are two parts of this table alike?

	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	11	13	15
3	0	3	6	12	15	21	24
4	0						
5							
6							

39. Multiply these base seven numbers.

(a) $\begin{array}{r} 52_{seven} \\ \underline{3} \end{array}$ (b) $\begin{array}{r} 34_{seven} \\ \underline{6} \end{array}$ (c) $\begin{array}{r} 421_{seven} \\ \underline{4} \end{array}$ (d) $\begin{array}{r} 21_{seven} \\ \underline{12}_{seven} \end{array}$

40. Here are some numbers in base seven. How would you write them in base ten?

(a) 43_{seven} (b) 526_{seven} (c) 304_{seven} (d) 260_{seven}

41. Change the numerals in the examples in Ex. 39 to numerals in base ten, and multiply again. Check your answers by changing the answers of Ex. 39 to base ten.

42. Explain the division of these base seven numbers. (The base is not written in the body of the work.)

$$\begin{array}{r} 454_{seven} \\ 6\overline{)4053}_{seven} \\ \underline{33}\phantom{_{seven}} \\ 45\phantom{_{seven}} \\ \underline{42}\phantom{_{seven}} \\ 33\phantom{_{seven}} \\ \underline{33}\phantom{_{seven}} \end{array}$$

(a) $6|\overline{42_{seven}}$ (b) $6|\overline{420_{seven}}$

(c) $6|\overline{435_{seven}}$

43. Divide 501_{seven} by 2_{seven}.

44. Divide 652_{seven} by 5_{seven}.

TESTS FOR DIVISIBILITY

45. Look at the counting chart for numbers written in base 10. How can you tell which numbers are divisible by 2? How can you tell which are divisible by 5? By 10?

46. Now look at the counting chart for numbers written in base seven, and copy the first ten numbers which are exactly divisible by 2. Is there an easy way to tell whether a number is divisible by 2 from its expressions in base seven?

47. From the counting chart in base seven, copy the first five numbers which are divisible by seven. How can you tell whether a number written in base seven is exactly divisible by seven?

48. (a) In the counting chart for numbers written in base ten, look at the ones which are divisible by nine. What do you notice about the sum of the digits of each numeral? Can you guess the general rule?

(b) Think of four numbers, larger than 100, which are divisible by 9. Add the digits in each numeral separately. Does the rule that you have just guessed work?

(c) Think of four numbers, larger than 100, which are not divisible by 9. See whether the rule you found works for those numbers.

(d) Can you tell how to decide whether a number is divisible by nine by adding the digits in its base ten numeral?

49. (a) In the counting chart for numbers written in base seven, look at the ones which are exactly divisible by six. Add the digits in each numeral. Do you notice a general rule?

(b) Think of four numbers larger than fifty, which are divisible by six. Write them in numerals in base seven, then add the digits in each numeral separately. Does your rule still work?

(c) Do the same for four numbers larger than fifty, which are not divisible by six. Does the rule work for these numbers?

(d) Can you tell how to decide whether a number is divisible by six by adding the digits in its base seven numeral?

50. Why should the test for divisibility by nine with numerals to base ten be like the test for divisibility by six with numerals to base seven?

51. (a) In the counting chart for base ten, choose five two-place numbers which are exactly divisible by 3. Find the sum of the digits for each number. Can you discover the general rule?

(b) Choose five numbers between ten and one hundred which are not exactly divisible by 3 and find the sum of the base ten digits. Does the rule still work?

(c) What seems to be a way to tell whether a number written in base ten is exactly divisible by 3?

(d) Look at the counting chart for base seven and choose five two-place numbers which are divisible by 3. Does the method you stated for numbers written in base ten seem to work for base seven numerals?

(e) Can you suggest another base for which this test works? If so, illustrate.

52. Write a number in base ten which can be exactly divided by 4. Write the same number in base seven. Can it be exactly divided by 4?

53. Think of a number greater than 5 which can be exactly divided by 5. Write the number in base seven numerals and in base ten numerals. Divide by 5, writing the quotient in base seven and also in base ten. Are the numerals the same? Do they represent the same number?

CHANGING FROM BASE TEN TO BASE SEVEN

You have learned how to change a number written in base seven numerals to base ten numerals. It is also easy to change from base ten to base seven. Let us see how that is done.

In base seven, the values of the places are 1, seven^1, seven^2, seven^3, and so on.

$$\text{seven}^1 = 7_{\text{ten}}$$
$$\text{seven}^2 = 49_{\text{ten}}$$
$$\text{seven}^3 = 343_{\text{ten}}$$

Example: Change 524_{ten} to base seven numerals.

Since 524 is larger than 343, first see how many $seven^3$ there are.

$$\begin{array}{r} 1 \\ 343\overline{)524} \\ \underline{343} \\ 181 \end{array}$$

The division shows there will be a 1 in the $seven^3$ position.

Now see how many 49's there are in 181.

$$\begin{array}{r} 3 \\ 49\overline{)181} \\ \underline{147} \\ 34 \end{array}$$

There will be a 3 in the $seven^2$ position.

Now see how many sevens there are in 34.

$$\begin{array}{r} 4 \\ 7\overline{)34} \\ \underline{28} \\ 6 \end{array}$$

There will be a 4 in the $seven^1$ position and 6 in ones' place.

So $524_{ten} = (1 \times 7^3) + (3 \times 7^2) + (4 \times 7^1) + (6 \times 1)$

$524_{ten} = 1346_{seven}$

54. Change 50_{ten} to base seven numerals.

55. Change 145_{ten} to base seven numerals.

56. Divide 1958_{ten} by ten. What is the quotient? What is the remainder? Divide the quotient by ten. What is the new quotient? Continue in the same way, dividing each quotient by 10, until you get a quotient of zero. How are the successive remainders related to the original number? Try the same process with $123,456,789_{ten}$. Try it with any other number.

57. Divide 524_{ten} by seven. What is the quotient? What is the remainder? Divide the quotient by seven, and continue as in Ex. 56, except that you divide by seven each time instead of ten. Compare the remainders with 524_{ten} written in base seven.

58. Can you give another method for changing from base ten to base seven?

OTHER BASES

You have studied numbers written in base ten and base seven. Now let us try some other bases.

59. How many symbols would there be in a system of notation in base five? base three? base four?

60. Draw sixteen x's on a paper. Draw lines around the groups you would use for base five. Then write the numeral showing the number of x's in base five. Be sure to write "five" after and below the numeral to show the base.

61. Now draw sixteen x's again, draw lines around groups of x's, and write the numeral in base four.

62. Draw sixteen x's again and show how to write the numeral in base three.

63. Make x's to show the numbers represented by

    (a) 13_{eight} (b) 23_{four} (c) 102_{three}

    (d) Do you have the same number of x's in all three groups of x's?

64. (a) How many threes are there in 20_{three}?

    (b) How many sixes are there in 20_{six}?

    (c) How many nines are there in 20_{nine}?

65. What is the smallest whole number which can be used as base for a system of number notation?

66. Here is part of a roll of tickets. Use different bases to record the number of tickets.

Number of Tickets	Base ten	Base six	Base four	Base three
one	1	1		
two	2	2		
three	3	3		
four	4	4		
five		5		
six		10		
seven				
eight				
nine				
ten				

67. 467_{eight} $(4 \times eight^2)$ $(6 \times eight^1)$ $(7 \times one)$

When we write the meaning of a numeral in this way, we say we are writing it in "expanded notation."

Write the following numbers in expanded notation.

(a) 638_{nine} (b) 245_{six} (c) 1002_{three}

68. How is each of these numbers written in base ten?

69. Write in expanded notation:

(a) 234_{five} (b) 103_{five} (c) 412_{five}

(d) If these numerals stand for amounts of money, what does each place represent?

Duodecimal Numerals

There are two bases of special interest. One of these is base twelve, which is the base of a system called the duodecimal system. We group many things by 12's, and call each group one dozen. We speak of a dozen eggs, a dozen rolls, a dozen pencils. When we have twelve twelves we call that one gross. Schools often buy pencils by the gross.

To write numbers in base twelve, you must have twelve symbols. You can make up symbols for ten and eleven, or use "t" and "e". The X's below are counted in base ten and in base twelve.

X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	Base ten
1	2	3	4	5	6	7	8	9	t	e	10	11	12	13	14	Base twelve

70. Write in expanded notation

(a) 146_{twelve} (b) $3t2_{\text{twelve}}$ (c) $47e_{\text{twelve}}$

Then express these numbers in base ten.

71. Many people believe that twelve is a better base for a system than ten is. See if you can find out why they think so.

Binary Numerals

The other base which has special importance is base two, called the binary system. Just as we use base ten and the Babylonians used base sixty, some modern high speed computing machines use base two, or the binary system.

Suppose we use two as base. What symbols will there be? Only 0 and 1.

72. Draw three x's, draw lines around groups of two, and write the number of x's in base two.

73. Make a counting chart in base two, for the numbers from zero to seventeen. Remember the places will stand for powers of two.

74. Make an addition chart for base two. How many addition facts are there?

75. Make a multiplication chart for base two. How many multiplication facts are there? Would they be hard to learn?

76. Below are some numbers expressed in base two. The first is

written to show the meaning of the digit in each place. Write the others in that way.

$$1011_{\text{two}} = (1 \times \text{two}^3) + (0 \times \text{two}^2) + (1 \times \text{two}^1) + (1)$$

(a) 111_{two}

(b) 1000_{two}

(c) 10101_{two}

(d) 11000_{two}

77. What number does 2^3 represent? 2^2? 2^1? Write the powers of two, up to two^9, in base ten.

78. What numbers are represented in Ex. 76? Give your answer in base ten notation.

79. Add these numbers which are expressed in binary notation.

(a) $\begin{array}{r} 101_{\text{two}} \\ \underline{10_{\text{two}}} \end{array}$ (b) $\begin{array}{r} 110_{\text{two}} \\ \underline{101_{\text{two}}} \end{array}$ (c) $\begin{array}{r} 10110_{\text{two}} \\ \underline{11011_{\text{two}}} \end{array}$ (d) $\begin{array}{r} 10111_{\text{two}} \\ \underline{11111_{\text{two}}} \end{array}$

Check by expressing the numbers in base ten and adding in the usual way.

80. Subtract these base two numbers.

(a) $\begin{array}{r} 111_{\text{two}} \\ \underline{101_{\text{two}}} \end{array}$ (b) $\begin{array}{r} 110_{\text{two}} \\ \underline{11_{\text{two}}} \end{array}$ (c) $\begin{array}{r} 1011_{\text{two}} \\ \underline{100_{\text{two}}} \end{array}$ (d) $\begin{array}{r} 11001_{\text{two}} \\ \underline{10110_{\text{two}}} \end{array}$

81. Check by expressing the numbers in base ten and subtracting in the usual way.

82. When people operate some kinds of high speed computing machines they usually express numbers in base two. Change these base ten numbers to base two.

(a) 35 (b) 128 (c) 12 (d) 100

83. If you have a peg board and some match sticks, you can represent base two numbers on the board. Leave a hole blank for 0 and put in a match stick for one. Represent two numbers

on the board, one below the other, and try adding on the board.

84. Now you have worked with several different number bases. Do some have special advantages?

85. Have you ever seen a weighing scale which uses weights for balances? You put the thing to be weighed on one side and enough weights to balance it on the other side. Then you add up the weights you have used to find the weight of the thing.

Suppose you want a set of weights which will make it possible for you to weigh a package of 1 pound, 2 pounds, 3 pounds, and so on up to 15 pounds (no fractions). What is the fewest number of weights you will need, and what must their weights be?

86. Here is a set of cards you can use to do a trick.

1	
1	9
3	11
5	13
7	15

2	
2	10
3	11
6	14
7	15

4	
4	12
5	13
6	14
7	15

8	
8	12
9	13
10	14
11	15

Tell a person to choose a number between 1 and 15, and to pick out the cards containing that number and give them to you. By adding the numbers at the top of the cards he gives you, you can tell him the number he chose.

See if you can figure out how the trick works. Then see if you can make a set of cards which you can use for numbers from 1 to 31.

87. Here is a method of multiplication, different from the one you use.

25 x 34 = ?

	25	34	
even	12	68	cross out
even	6	136	cross out
	3	272	
	1	544	

Divide the numbers in the first column by 2, throwing away any remainders, and multiply the numbers in the second column by 2. Then cross out the numbers in the second column which are opposite even numbers in the first column, and add the numbers in the second column which are left. Does this give the correct product for 25 x 34? Try to figure out why this method works. (Remember base 2.)

88. Do you have an abacus in your classroom? If not, try to borrow one from a primary room or make one. Then make one to use for numbers expressed in base two.

UNIT III

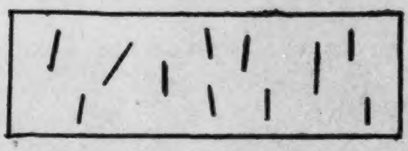
NATURAL NUMBERS AND ZERO

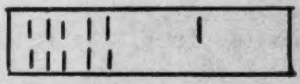
We Learned to Count

We use numbers in counting. We call them natural numbers. In Don's family there are five children. Jane buys twelve oranges at the store. There are five thousand four hundred eighty-five people in the stadium. We have learned that numerals are names of numbers. If we are using our usual system of numeration, then the numerals 5, 12, 5485 are the names of the numbers mentioned. Man invented numerals when he recognized numbers in the world around him.

We know that there is the same number of people and noses in the room. If there are 50 stars on the flag, then there are 50 states. There are 36 pupils in the room and the teacher has written the name of each pupil on a small card. The number of cards and the number of pupils is the same, assuming that the teacher has no blank cards. The number words, which we use form a standard set which we memorize in a certain order, and match with any set that we wish.

We may have learned about numbers in different ways. Some of us may have learned number names and then how to count. We learned that the last name we used gave us the number of the set counted. For example, here we have a collection or set of marks. We ask how many are there. By counting, we find that the last name we used in the matching process of a name with a mark was eleven. So we say the number of marks is eleven. If these are arranged in some way to form a pattern we may recognize a pattern of eleven without counting each mark. And that was

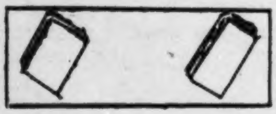


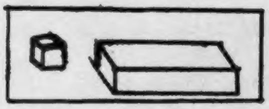


Is this easier? Do you need to count?

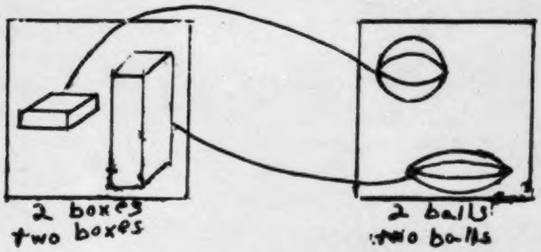
the way we learned to recognize groups without counting. We just knew how many were there. Many of us can recognize the number of a set of things without counting -- unless they are so mixed up that we can't see a familiar pattern.

Others of us may have learned the number of a set first without counting. We were told that here is a picture of two books and of two boxes.



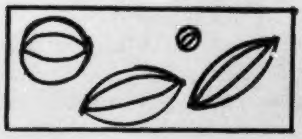


Perhaps we wondered what it was that was two but somehow we saw it. Things may have many properties in common -- like size, color, shape, use, taste. The same may be said of collections or sets of things. They have the property of containing the same number of members. This "sameness" is called number. Any time two sets of things can be matched like this --

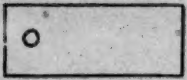


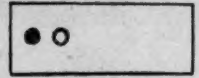
the two sets have a "same as" relationship. They both have the same number property. We use the same numeral to describe this number property.

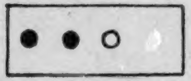
We know that this is not a picture of 3 balls. We call the number of this set "four." We learned number names for different collections. We found that we could arrange these in an

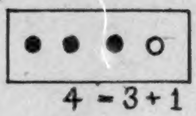


order, each number being just one more than the one before it


1


$2 = 1 + 1$


$3 = 2 + 1$


$4 = 3 + 1$

starting with 1, 1 and 1 more is 2, 2 and 1 more is 3, 3 and 1 more is 4. It was then that we soon learned to group by tens, place value, and a system of numeration.

What would it have been like if we had learned a completely different name for each number?

We can, of course, count the number of units in two sets, provided that there is a term that applies to the objects in both sets. For instance, two cows and three horses make a set of five animals. Also three c's and two b's are five letters.

Exercises - 1

1. Which of the following are natural numbers:

   (a) 1 (b) 7 (c) \$3 (d) .6 (e) 20 (f) 1/3

   (g) 3/3 (h) 100 (i) 8/4

2. Rearrange these natural numbers in their usual order.

   (a) 1, 2, 3, 6, 4, 5

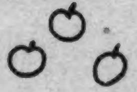
   (b) 1 + 1, 2 + 1, 1 + 3, 0 + 5, 1 + 6, 5 + 1, 5 + 3

   (c) IV, V, XI, VI, VII, VIII, X, IX

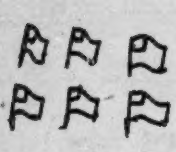
3. Which of the following numbers are natural numbers not between one and ten? 2, 5, 7, 8. Not between six and eleven? 4, 9, 11, 15. Not between one and fifty? 1, 15, 25, 28, 40. Not between one and ten? 13, 14, 22, 78, 86.

4. Here are sets of objects arranged so that the number of objects in the set may be recognised without counting. Write the name of the number of objects in each set.

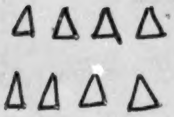
a)



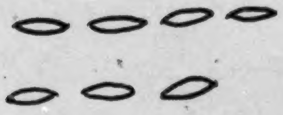
b)



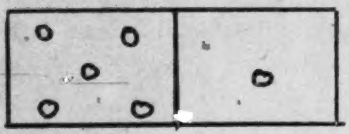
c)



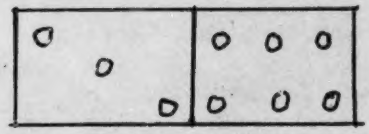
d)



e)

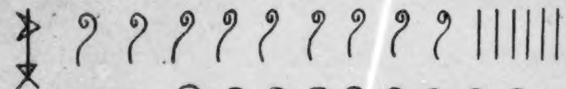


f)

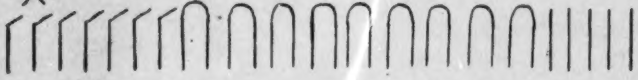


5. Rewrite the following Egyptian numerals, so that they may be easily read without counting

(a)



(b)

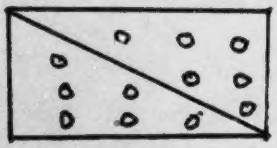


6. Show a grouping of marks which could easily be read without counting for the following

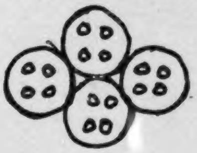
(a) 10 (b) 12 (c) 16 (d) 25

7. Tell how many dots are in the figures without counting each dot.

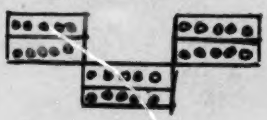
a)



b)



c)



8. Some people are standing in a room, in which there is a set of chairs. You want to find out if more chairs are needed. Is it necessary to count the people and the chairs to find out? What can you do to get quickly the needed information? Do you then know how many people are in the room?

9. A theater owner wants to know how many people attended the show last night. He knows the first ticket sold was numbered 60588 and the last ticket sold was 60735. Does he need to hire a man

to count the people as they come in? How many people attended the theater for that show?

10. A teacher has graded homework papers and recorded the grades in his gradebook. How can he quickly check to see if each student handed in his homework?

11. Tell how many more symbols are in one set than in the other. Do not count each set.

    (a) ○ ○ ○ ○ ○ ○
    X X X X

    (b) □ □ □ □ □ □ □
    △ △ △ △

12. Make two sets of different figures and group them to show that one set is 5 more than the other.

13. When you count the number of people in the room, does it matter in what order you count? What must you be careful about in counting?

14. Sometimes we say that there are twice as many of one kind of thing as another. Give some examples in which you would know that there are twice as many of one kind as another without counting either set.

The Commutative Property

If you have two boxes of pencils and in one of them there are 5 pencils and in the other there are two pencils, how many pencils do you have? What do you do to answer this question? If you say you "add" do you think, 5 + 2 or 2 + 5?

The arithmetic teacher read two large numbers to be added. John did not understand what his teacher said when she read the first number. He wrote the second number when she read it. Then he asked her to repeat the first number. When she read it again, John wrote it on his paper below the second number instead of above it. Will John find the same sum as the other students who heard all the dictation the first time? We say the sum of CCLXIV and CXXXII is the same as the sum of CXXXII and CCLXIV. In the binary system we find that the sum of 1101011 and 100111 is the same as the sum of 100111 and 1101011. Is this right?

Maybe you were told in elementar school that a good way to discover a mistake made in addition problems is to add again in "the other direction." If you "added down" then you might check your problem by "adding up." Even before that we learned that to put 3 balls with 2 balls gave us the same number of balls as putting 2 balls with 3 balls.

You have just recognised the commutative property for addition of natural numbers. It means that the order in which we add two numbers does not make any difference in the sum of the two numbers.

3 added to 4 is 7 or $4 + 3 = 7$

4 added to 3 is 7 or $3 + 4 = 7$

Both are other names for 7. We can write

$$4 + 3 = 3 + 4.$$

The statement of this law in words is quite clumsy. It is simpler and clearer to say this in mathematical language:

"If a and b are natural numbers, then $a + b = b + a$."

We have just discussed the case in which $a = 4$ and $b = 3$.

Let us suppose Don's patrol goes on a trip. Twenty-nine boys go, each taking 3 dimes and 8 pennies for food. How many cents can be spent for food? How do you find the answer to this problem?

This way? $\begin{array}{r} 29 \\ \times\ \underline{38} \end{array}$ or this way? $\begin{array}{r} 38 \\ \times\ \underline{29} \end{array}$

Bill's homeroom has a party. 38 boys and girls came to the party. They shared the cost of the party and each was asked to pay 29 cents. How much did the party cost? Do you find the answer

This way? $\begin{array}{r} 38 \\ \times \underline{29} \end{array}$ or this way? $\begin{array}{r} 29 \\ \times \underline{38} \end{array}$

Suppose we have 5 rows of chairs with 3 in each row. We decided to change the arrangement to make 3 rows of chairs with 5 in each row. Do we need more chairs?

x x x
x x x
x x x
x x x
x x x

(5 x 3)

x x x x x
x x x x x
x x x x x

(3 x 5)

The product of two natural numbers is the same, whether the first be multiplied by the second or the second be multiplied by the first. This statement is called the commutative property for multiplication of natural numbers. It means that it makes no difference which number is the multiplier and which is the multiplicand. This statement may seem clumsy to you. Can you state this property in symbols?

We can use this idea to detect mistakes we might make in multiplying one number by another. We found these products.

$$\begin{array}{r} 3927 \\ \underline{\times\ 485} \\ 15635 \\ 31416 \\ \underline{15708} \\ 1899595 \end{array} \qquad\qquad \begin{array}{r} 485 \\ \underline{\times\ 3927} \\ 3395 \\ 970 \\ 4365 \\ \underline{1455} \\ 1904595 \end{array}$$

As an application of the commutative property, we realize we have made a mistake. Find the mistake.

The idea of using letters to stand for any number whatsoever in stating general principles of arithmetic is a very useful part of mathematical language. There is a danger that the letter "x" and the multiplication sign may be mistaken for each other, so we frequently use a dot for multiplication. For example we can write "4 · 3" for "4 x 3", "a · b" for "a x b."

In the Exercises the symbol "<" is read "less than". For example:

$$5 + 3 < 5 + 4$$

is read "5 + 3 is less than 5 + 4." Can you tell what the symbols ">" and "≠" might mean?

Exercises - 2

A

1. (a) Is 6 + 4 equal to 4 + 6?
   (b) Is 14 + 7 equal to 7 + 14?
   (c) Is 35 + 64 equal to 64 + 35?
   (d) Is 315 + 462 equal to 462 + 315?
   (e) Is 315 + 462 equal to 264 + 513?
   (f) Is 475 + 381 equal to 183 + 574?
   (g) Is $(13 + 32)_{four}$ equal to $(32 + 13)_{four}$?
   (h) Is 5 + 3 < 3 + 5?
   (i) Is 5 + 2 < 3 + 5?
   (j) Is 5 + 3 < 2 + 5?
   (k) Is 5 + 4 < 3 + 5?

B

1. (a) Is 72 + 31 equal to 31 + 72?
   (b) Is 16 + 52 equal to 25 + 61?
   (c) Is 58 + 94 equal to 94 + 58?
   (d) Is 465 + 332 equal to 564 + 233?
   (e) Is 735 + 254 equal to 537 + 452?
   (f) Is 851 + 367 equal to 158 + 763?
   (g) Is 58 + 94 equal to 94 + 58 when these are written in the base twelve?

   Insert a symbol which makes the following true statements.

   (h) 7 + 4   4 + 7
   (i) 9 + 6   7 + 9
   (j) 12 + 5   5 + 11
   (k) 46 + 81   64 + 18

(l) Is $5 + 3 < 4 + 5$?

(m) Is $5 + 3 < 3 + 6$?

(n) Is $6 + 3 < 4 + 5$?

2. Add and use the commutative property to check addition

$$\begin{array}{r} 465 \\ \underline{179} \end{array} \qquad \begin{array}{r} 37{,}461 \\ \underline{73{,}135} \end{array}$$

$$\begin{array}{r} 73967 \\ \underline{81785} \end{array} \qquad \begin{array}{r} (43)_5 \\ \underline{(32)_5} \end{array}$$

3. (a) Is 6 x 5 equal to 5 x 6?

   (b) Is 7 x 4 equal to 4 x 7?

   (c) Is $(3 + 2) \cdot (5 + 2)$ equal to $(5 + 2) \cdot (3 + 2)$?

   (d) Is $(3 + 1) \cdot (2 + 3)$ equal to $(4 + 1) \cdot (2 + 2)$?

   (e) Is 24 x 43 equal to 43 x 24?

4. Multiply and use the commutative property to check multiplication

$$\begin{array}{r} 36 \\ \times\ \underline{57} \end{array} \qquad \begin{array}{r} 305 \\ \times\ \underline{84} \end{array} \qquad \begin{array}{r} 476 \\ \underline{609} \end{array}$$

$$\begin{array}{r} 7604 \\ \times\ \underline{1008} \end{array} \qquad \begin{array}{r} (34)_8 \\ \times\ \underline{(7)_8} \end{array}$$

Which of the following are true?

5. (a) $4 + 6 = 6 + 4$

   (b) $7 - 2 = 2 - 7$

   (c) $3 \times 9 = 9 \times 3$

   (d) $41 \times 16 = 16 \times 41$

(l) $39 + 72$ $\quad$ $39 + 72$

(m) $35 + 53$ $\quad$ $53 + 35$

(n) $47 + 85$ $\quad$ $81 + 51$

2. Add and use the commutative property to check addition

$$\begin{array}{r} 5{,}769 \\ \underline{1{,}785} \end{array} \qquad \begin{array}{r} 465{,}781 \\ \underline{947{,}977} \end{array}$$

$$\begin{array}{r} (10111)_2 \\ \underline{(10101)_2} \end{array} \qquad \begin{array}{r} (4t7e)_{12} \\ \underline{(124t)_{12}} \end{array}$$

3. (a) Is 9 x 7 equal to 7 x 9?

   (b) Is 43 x 56 equal to 56 x 43?

   (c) Is $(6 + 1) \cdot (4 + 5)$ equal to $(4 + 5) \cdot (6 + 1)$?

   (d) Is $(9 + 2) \cdot (7 + 8)$ equal to $(9 + 6) \cdot (5 + 6)$?

   (e) Is $(486) \cdot (501)$ less than $(501) \cdot (486)$?

4. Multiply and use the commutative property to check multiplication

$$\begin{array}{r} 76 \\ \times\ \underline{98} \end{array} \qquad \begin{array}{r} 47503 \\ \times\ \underline{20057} \end{array}$$

$$\begin{array}{r} (241)_5 \\ \times\ \underline{(32)_5} \end{array} \qquad \begin{array}{r} (1011)_2 \\ \times\ \underline{(110)_2} \end{array} \qquad \begin{array}{r} (465)_{12} \\ \times\ \underline{(372)_{12}} \end{array}$$

Which of the following are true?

5. (a) $(37 \times t)_{12} = (t \times 37)_{12}$

   (b) $(31 - 7) = (7 - 31)$

   (c) IV - I = V

   (d) $49 \times 63 = 64 \times 48$

(e) 8 ÷ 2 = 2 ÷ 8

(f) 65 - 47 = 47 - 65

(g) 72 ÷ 12 = 12 ÷ 72

Insert a symbol to make the following true

6. (a) 25 + 17   17 + 25

(b) 26 - 4   4 - 26

(c) 60 ÷ 20   20 ÷ 60

(d) 32 x 12   12 x 32

(e) 7 ÷ 84   84 ÷ 7

(e) 78 + 54 = 53 + 79

(f) 65 ÷ 5 = 5 ÷ 65

(g) 72 ÷ 9 = 9 ÷ 72

Insert a symbol to make the following true

6. (a) 47 + 182   182 + 47

(b) 71 - 56   56 - 71

(c) 625 ÷ 25   25 ÷ 625

(d) 76 x 67   67 x 76

(e) 25 ÷ 575   575 x 25

Addition and subtraction are operations which can be performed on numbers. They are examples of binary operations, since they are performed on a pair of numbers. What other binary operations are frequently used in arithmetic?

The operations of addition and multiplication each have the property that the commutative property holds. Notice the similarity between the equations

$$a + b = b + a,$$

and

$$a \cdot b = b \cdot a.$$

We say that these operations are commutative. To find whether subtraction is commutative we must ask whether $a - b = b - a$ is true for any numbers a and b. Is it true for $a = 5$ and $b = 5$? Is division a commutative operation? Give an example illustrating your answer.

Exercises - 2a

7. Which of the following involve commutative operations?

(a) 1 + 2

(b) 6 + 8

(c) 7 • 9

(d) 4 - 5

(e) 12 ÷ 3

(f) 9 - 4

8. Which of the following activities are commutative?
   (a) To put on a hat and then a coat
   (b) To walk down the block and cross the street
   (c) To pour red paint into blue paint
   (d) To pull up a chair and sit on it
   (e) To walk through a doorway and close the door.

9. What operations and activities can you list which are commutative? Which are not commutative? (You need not list more than five for each.)

10. Suppose we define a new operation, symbolized by $\vee$, like this:

$$a \vee b = a + b + (a \cdot b)$$

Compute $3 \vee 4$ and $4 \vee 3$. Is this operation commutative?

The Associative Property

How do you add $\begin{array}{r}12\\ \underline{+3}\end{array}$? You think $2 + 3 = 5$, and bring down the 1. What have you actually done? Why does it work? You know that $12 = 10 + 2$. Your problem $\begin{array}{r}12\\ \underline{+3}\end{array}$, is $(10 + 2) + 3$. You actually found $10 + (2 + 3)$. We know $12 + 3 = 10 + (2 + 3)$. Let us try some other numbers.

What do we mean by $1 + 2 + 3$? Do we mean $(1 + 2) + 3$, or do we mean $1 + (2 + 3)$? Does it make any difference? We have said the order we add two numbers doesn't make any difference. Now we see that the way we group numbers to add them doesn't change the sum.

$$1 + (2 + 3) = 1 + 5 = 6$$
$$(1 + 2) + 3 = 3 + 3 = 6$$

We call this idea the associative property of addition for natural numbers. Using both the commutative and associative properties of addition most anything goes in adding numbers. We can mix them up in just about any way we choose and still get the same answer. Maybe we do not stop to think about how we mix these up. For example,

> When asked to add 12 + 14 + 18, John thought, "12 + 14 = 26, and 26 + 18 = 44." This expression represents John's procedure -- (12 + 14) + 18. Bill thought, "12 + 18 is 30, and 30 + 14 = 44." His thinking might be represented by (12 + 18) + 14. How did Bill know that (12 + 14) + 18 = (12 + 18) + 14?
>
> (12 + 14) + 18 = 12 + (14 + 18) By the associative property of addition
>
> 12 + (14 + 18) = 12 + (18 + 14) By the commutative property of addition
>
> 12 + (18 + 14) = (12 + 18) + 14 By the associative property of addition

The commutative principle means we may change the order of any two numbers we are adding without changing the sum, that is a + b = b + a. The associative principle means that no matter how we may group numbers for purposes of adding numbers, the result is the same, that is, (a + b) + c = a + (b + c).

When you multiply 2 • (30) you actually compute (2 • 3) • 10. You know 30 = 3 • 10. Is 2(3 • 10) = (2 • 3) • 10?

The associative principle holds for multiplication with natural numbers. What do we mean by 2 x 5 x 4? Let's try some ways.

> (2 x 5) x 4 = 10 x 4 = 40
>
> 2 x (5 x 4) = 2 x 20 = 40

Both give the same answer. We may group the factors of a product any way we please without changing the product. This is called the associative property of multiplication for natural numbers, that is, $(a \times b) \times c = a \times (b \times c)$.

Since addition and multiplication are associative, there is no possibility of confusion when we write $1 + 2 + 3$ or $1 \cdot 2 \cdot 3$ omitting the parentheses. This is an example of mathematical slang which is allowed since it does not lead to any confusion.

However in $2 + 3 \cdot 4$ it does make a difference how we group the numbers since

$$(2 + 3) \cdot 4 = 5 \cdot 4 = 20$$

but

$$2 + (3 \cdot 4) = 2 + 12 = 14$$

Therefore it is wrong to omit the parentheses here and "$2 + 3 \cdot 4$" is nonsense unless we make some agreement about its meaning.

Exercises - 3

1. Show that the following are true.

Example: $(4 + 3) + 2 = 4 + (3 + 2)$

$$7 + 2 = 4 + 5$$

$$9 = 9$$

A	B
(a) $(4 + 7) + 2 = 4 + (7 + 2)$	(a) $(21 + 5) + 4 = 21 + (5 + 4)$
(b) $8 + (6 + 3) = (8 + 6) + 3$	(b) $(34 + 17) + 29 = 34 + (17 + 29)$
(c) $46 + (73 + 98) = (46 + 73) + 98$	(c) $436 + (476 + 1) = (436 + 476) + 1$
(d) $(6 \times 5) \times 9 = 6 \times (5 \times 9)$	(d) $(9 \times 7) \times 8 = 9 \times (7 \times 8)$
(e) $(24 \times 36) \times 20 = 24 \times (36 \times 20)$	(e) $(57 \times 80) \times 75 = 57 \times (80 \times 75)$

2. We now know that addition and multiplication of natural numbers

have the associative property. Let's look at subtraction. Is it associative? If it were for any natural numbers a, b, and c, $(a - b) - c = a - (b - c)$. Here is an example, try it.

Does $(10 - 7) - 2 = 10 - (7 - 2)$?

Does $(18 - 5) - 3 = 18 - (5 - 3)$?

Does subtraction have the associative property?

3. If division were associative, this would mean for any natural numbers, a, b, and c, $(a \div b) \div c = a \div (b \div c)$. Test this with this example: $(32 \div 8) \div 2 = 32 \div (8 \div 2)$. What conclusion do you come to? Make up another example to show again that you are right.

4. Rewrite these problems using the associative and commutative properties when necessary to make the addition easier. Use parentheses to show which additions are done first.

A	B
(a) $6 + 1 + 9$	(a) $72 + 90 + 10$
(b) $5 + 7 + 2$	(b) $50 + 36 + 20$
(c) $63 + 75 + 25$	(c) $28 + 75 + 25$
(d) $26 + 72 + 4$	(d) $83 + 46 + 17$
(e) $340 + 522 + 60$	(e) $3 + 5 + 7 + 15$
(f) $45 + 15 + 63$	(f) $56 + 23 + 44 + 77$
(g) $13 + 36 + 4$	(g) $18 + 16 + 24 + 2$

5. Rewrite these using the associative and commutative properties when necessary to make the multiplication easier. Use parentheses to show which multiplications are done first.

A	B
(a) $13 \times 10 \times 2$	(a) $2 \times 67 \times 5$
(b) $5 \times 45 \times 2$	(b) $25 \times 4 \times 86$
(c) $7 \times 25 \times 4$	(c) $38 \times 50 \times 2$
(d) $50 \times 2 \times 33$	(d) $3 \times 11 \times 4$

6. Is the following statement true? To multiply 2 by the product of 5 and 6 we can multiply 6 by the product of 2 and 5. Explain.
7. Is the following statement true? In order to double the product of 6 and 5, you double 6, double 5 and take the product of the doubles. Use parentheses to show what is being done. Explain the reasons for your answer.
8. Is the following statement true? In order to double the sum of 6 and 5, you double 6, double 5 and take the sum of the doubles. Use parentheses to show what is being done. Can you explain the reasons for your answer from what you have studied so far?
9. Make the symbols $2 + 3 \cdot 4 \div 2$ meaningful by grouping the numbers. In how many different ways may this be done?

The Distributive Property

Eight girls and four boys -- twelve children altogether -- are planning a skating party. For a merrier party, each girl invites another girl and each boy invites another boy. The number of girls has been doubled. The number of boys has been doubled. Has the number of children been multiplied by two? by four? by twelve?

Altogether, there will be $(2 \cdot 8)$ girls and $(2 \cdot 4)$ boys or a total of $(2 \cdot 8) + (2 \cdot 4)$ children at the party. When the party was planned, there were $(8 + 4)$ children. The final number of children is the product of 2 and $(8 + 4)$. We see that --

$$(2 \cdot 8) + (2 \cdot 4) = 16 + 8 = 24$$

$$2 \cdot (8 + 4) = 2 \cdot 12 = 24$$

So, we can write

$$(2 \cdot 8) + (2 \cdot 4) = 2 \cdot (8 + 4)$$

You have been using this property since the third grade. Take $\begin{array}{r}12\\ \underline{\times 3}\end{array}$

What do you actually think? You say to yourself, "$3 \cdot 2 = 6$, $3 \times 1 = 3$, and the product is 36." Your method is correct because

$$3(10 + 2) = (3 \cdot 10) + (3 \cdot 2)$$

By the commutative principle it is also true that

$$(10 + 2)3 = (10 \cdot 3) + (2 \cdot 3)$$

as we see when we change the order of multiplication.

This idea is familiar to us. We learned that $\begin{array}{r}375\\ \underline{\times\ \ 7}\end{array}$ is the same as $(7 \cdot 5) + (7 \cdot 70) + (7 \cdot 300)$. But instead of writing 375 as $(300 + 70 + 5)$ let's write 375 as $(145 + 230)$. Is $(7 \cdot 375)$ the same as $7 \cdot (145 + 230)$?

We can write $\begin{array}{r}83\\ \underline{\times 45}\end{array}$ as $(80 + 3) \cdot 45 = (80 \cdot 45) + (3 \cdot 45)$. This may be more familiar to some of us if we wrote $\begin{array}{r}45\\ \underline{\times\ 3}\end{array}$ and $\begin{array}{r}45\\ \underline{\times 80}\end{array}$.

In the more usual form we have

$$\begin{array}{r}45\\ \underline{\times\ 83}\\ 135\\ \underline{360\ \ }\\ 3735\end{array}$$

We recognise in the product that $45 \cdot 3 = 135$ and $45 \cdot 80 = 3600$.

This idea that we have been describing links together the operations of multiplication and addition. We refer to this idea as the distributive principle of multiplication over addition.

If we use letters to represent numbers, we can say

$a(b + c) = (a \cdot b) + (a \cdot c)$ or $(b + c) \cdot a = (b \cdot a) + (c \cdot a) = (a \cdot b) + (a \cdot c.)$

Exercises - 4

1. **Find simpler names**

A	B
(a) $(3 \cdot 9) + 6$	(a) $(4 \cdot 9) + 16$
(b) $3 \cdot (9 + 6)$	(b) $7(3 + 9)$
(c) $5 \cdot (2 + 7)$	(c) $(7 \cdot 3) + 9$
(d) $(5 \cdot 2) + 7$	(d) $(8 \cdot 6) + (3 \cdot 7)$
(e) $(8 \cdot 4) + (2 \cdot 3)$	(e) $(6 + 5) + (7 \cdot 9)$
(f) $(2 + 3) + (6 \cdot 8)$	(f) $(8 + 7) \cdot (9 \cdot 5)$
(g) $(3 + 9) \cdot (2 \cdot 3)$	(g) $(46 - 17) + 17$
(h) $(14 - 7) + 3$	(h) $(7 \cdot 3) + (7 \cdot 9)$
(i) $(7 \cdot 6) + (7 \cdot 9)$	(i) $16(7 + 6)$
(j) $7 \cdot (6 + 9)$	(j) $(16 \cdot 7) + (16 \cdot 6)$

2. **Show that the following are true**

Example: $3(4 + 3) = (3 \cdot 4) + (3 \cdot 3)$

$3 \cdot 7 \qquad 12 + 9$

$21 \qquad 21$

A	B
(a) $4(7 + 5) = (4 \cdot 7) + (4 \cdot 5)$	(a) $9(7 + 5) = (9 \cdot 7) + (5 \cdot 9)$
(b) $(3 \cdot 4) + (3 \cdot 8) = 3 \cdot (4 + 8)$	(b) $(11 \cdot 3) + (11 \cdot 4) = 11(3 + 4)$
(c) $(5 \cdot 2) + (5 \cdot 3) = 5(2 + 3)$	(c) $12(5 + 6) = (12 \cdot 5) + (6 \cdot 12)$
(d) $(6 \cdot 3) + (6 \cdot 2) = 6(3 + 2)$	(d) $(15 \cdot 6) + (15 \cdot 5) = 15(6 + 5)$
(e) $7(9 + 8) = (7 \cdot 9) + (7 \cdot 8)$	(e) $23(2 + 3) = (23 \cdot 2) + (23 \cdot 3)$
(f) $2(16 + 8) = (2 \cdot 16) + (2 \cdot 8)$	(f) $(3 \cdot 99) + (5 \cdot 99) = 99(3 + 5)$
(g) $(3 \cdot 6) + (4 \cdot 6) = 6(3 + 4)$	(g) $128(10 + 20) = (128 \cdot 10) + (128 \cdot 20)$
(h) $9(1 + 2) = (9 \cdot 1) + (9 \cdot 2)$	(h) $(1000 \cdot 10) + (1000 \cdot 20) = 1000(10 + 20)$

(i) $(10 \cdot 6) + (3 \cdot 10) = 10\ (6 + 3)$

(j) $(6 \cdot 8) + (6 \cdot 7) = 6(8 + 7)$

(i) $(8 \cdot 7) + (8 \cdot 7) = 8(7 + 7) = 7(8 + 8)$

(j) $15(3 + 3) = 3(15 + 15)$

3. **Insert symbols to make each a true sentence.**

(a) $3(4 + 3) \quad (3 \cdot 4) + (3 \cdot 3)$

(b) $2(4 \quad 5) = (2 \cdot 4) + (2 \cdot 5)$

(c) $(5 \cdot 2) + (5 \quad 3) = 5(2 + 3)$

(d) $7(3 + 4) \quad (7 \cdot 3) + (7 \quad 4)$

(e) $(2 \cdot 7) + (3 \cdot \quad) = 7(2 + 3)$

(f) $(3 \cdot 5) + (3 \cdot 4) = 3(5 \quad)$

(a) $3(6 + 4) = (3 \cdot \quad) + (3 \cdot \quad)$

(b) $7(2 + \quad) = (7 \cdot \quad) + (\quad \cdot 3)$

(c) $(\quad \cdot 2) + (\quad \cdot 5) = 8(\quad\quad)$

(d) $(\quad \cdot 4) + (\quad \cdot 4) = \quad (6 + 7)$

(e) $11(\quad\quad) = (\quad \cdot 2) + (\quad \cdot 3)$

(f) $8(5 \quad 6) = (\quad 5 \quad) + (\quad\quad)$

4. **Using the distributive property rewrite each of the following:**

Examples: (a) $5(2 + 3) = (5 \cdot 2) + (5 \cdot 3)$

(b) $(6 \cdot 4) + (6 \cdot 3) = 6(4 + 3)$

(a) $4(2 + 3)$

(b) $7(4 + 6)$

(c) $(9 \cdot 8) + (9 \cdot 2)$

(d) $6(13 + 27)$

(e) $(12 \cdot 5) + (12 \cdot 7)$

(a) $(5 \cdot 26) + (5 \cdot 7)$

(b) $8(14 + 17)$

(c) $7(213 + 787)$

(d) $(27 \cdot 13) + (27 \cdot 11)$

(e) $(18 \cdot 19) + (17 \cdot 18)$

5. **Try these:** Using the idea of the distributive property we can write

$36 + 42$ as $(6 \cdot 6) + (6 \cdot 7) = 6 \cdot (6 + 7)$

$18 + 15$ as $(6 \cdot 3) + (5 \cdot 3) = 3(6 + 5)$

Can you rewrite these in the same way?

Check your work to see that you are right.

Example: $18 + 15 = (6 \cdot 3) + (5 \cdot 3) = 3(6 + 5)$

33	18 + 15	3 x 11
	33	33

	A		B
(a)	(6 + 4)	(a)	(8 + 12)
(b)	(12 + 9)	(b)	(14 + 21)
(c)	(10 + 15)	(c)	(36 + 18)
(d)	(24 + 18)	(d)	(40 + 16)
(e)	(28 + 32)	(e)	(12 + 48)
(f)	(21 + 14)	(f)	(56 + 42)
(g)	(25 + 15)	(g)	(72 + 27)
(h)	(3 + 6)	(h)	(7 + 63)

6. Using the idea of the distributive principle, we can write 45 as $(40 + 5)$ and 23 as $(20 + 3)$. Then the product would be

$45 \cdot 23 = (40 + 5) \cdot (20 + 3) = 40(20 + 3) + 5(20 + 3) =$

$(40 \cdot 20) + (40 \cdot 3) + (5 \cdot 20) + (5 \cdot 3) = 800 + 120 + 100 + 15 = 1035$

Check this result by multiplication of 45 and 23. Rewrite the following products in the same way, and check the results.

	A	B
(a)	$78 \cdot 45$	$86 \cdot 34$
(b)	$13 \cdot 76$	$53 \cdot 19$
(c)	$567 \cdot 84$	$623 \cdot 72$

7. Which of the following are true?

(a) $3 + 4 \cdot 2 = (3 + 4) \cdot (3 + 2)$

(b) $3 \cdot (4 - 2) = (3 \cdot 4) - (3 \cdot 2)$

(c) $(4 + 6) \cdot 2 = (4 \cdot 2) + (6 \cdot 2)$

(d) $(4 + 6) \div 2 = (4 \div 2) + (6 \div 2)$

8. Suppose we introduct the new symbol "$\wedge$" by the definition $a \wedge b = b^a$. For example, $2 \wedge 3 = 3^2 = 9$

$3 \wedge 2 = 2^3 = 8$.

Which of the following are true?

(a) $2 \wedge (3 \cdot 4) = (2 \wedge 3) \cdot (2 \wedge 4)$

(b) $(3 + 4) \wedge 2 = (3 \wedge 2) + (4 \wedge 2)$

(c) $2 \wedge (12 \div 3) = (2 \wedge 12) \div (2 \wedge 3)$

The Closure Property

There is another property of natural numbers associated with the idea of addition and multiplication. If we add any two natural numbers (they may be the same number), our answer is also a natural number. For example,

$4 + 7 = 11$ and only 11

$5 + 5 = 10$ and only 10

$85 + 91 = 176$ and only 176

All are natural numbers.

We say that the set of natural numbers is closed with respect to addition, or closed under addition. That is, if we add any two natural numbers, we shall always get one and only one natural number as their sum.

The same thing is true for multiplication. The product of two natural numbers is one and only one natural number. For example, $2 \times 8 = 16$ and only 16. It isn't ever some other number like 38, 51, etc.

The operations of addition and multiplication have the property that when either is applied to a pair of natural numbers, in a given order, the result is a uniquely determined natural number.

We can say the set of natural numbers from one to ten is not closed under addition. $(2 + 4)$ is a number less than 10 but is $(6 + 5)$ less than or equal to 10?

Is the set of natural numbers closed with respect to division? If we find the quotient of 8 and 2 or $8 \div 2$, we get another natural number. But if we try $9 \div 2$, then we do not get a natural number for an answer. We say that the set of natural numbers is not closed with respect to division. Too, we cannot always subtract one natural number from another and get a natural number. $16 - 4 = 12$. All of these are natural numbers. But what about $4 - 15$? Can our answer be a natural number? We say that this problem is impossible to solve if we must have a natural number as an answer.

Exercises - 5

1. Is the sum of two odd numbers always an odd number?
   Is the set of odd numbers closed under addition?
2. Is the set of even numbers closed under addition?
3. Is the set of all multiples of 5 (5, 10, 15, 20, etc.) closed with respect to addition?
4. What is true of the sets of numbers in Exercises 1, 2, and 3 under multiplication?
5. Are the following sets of numbers closed with respect to addition?
   (a) Set of natural numbers greater than 50?
   (b) Set of natural numbers from 100 through 999?
   (c) Set of natural numbers less than 48?
   (d) Set of natural numbers ending in 0?

6. Are the sets of numbers in Exercise 5 closed with respect to multiplication?
7. Are all sets of natural numbers which are closed with respect to addition also closed with respect to multiplication? Why?
8. Are any of the sets of numbers in Exercise 5 closed under subtraction?
9. Are any of the sets of numbers in Exercise 5 closed under division?

Inverse Operations

Often we do something and then we undo it. We open the door; we shut it. We turn on a light; we turn it off. We put on our coats; we take them off. We put two sets of things together into one set; we separate one set of things into two sets.

We call subtraction the inverse operation to addition. The inverse of adding 5 is subtracting 5.

What does the grocer do when you buy something for 31 cents, and pay for it with a dollar bill? Does he say, "100 - 31 = 69, here is 69 cents change."? No, he does not even mention the number 69. He counts out money into your hand, "32, 33, 34, 35, 40, 50, one dollar." He finds out how much to add to 31 in order to have 100. He answers the question:

$$31 + ? = 100$$

Can you subtract 23 from 58 by adding?

$$23 + ? = 58$$

You might think, " 3 + 5 = 8, 2 + 3 = 5, and the missing number is 35."

You have used addition to check subtraction. Can you use multiplication to check division?

We call division the inverse operation to multiplication. The inverse of multiplying by 5 is dividing by 5. The operation of multiplying 2 by 4 gives 8: 4 x 2 = 8. Now what can we do to 8 with 4 to give us 2? We divide. To describe the operation we can write 8/4 = 2 because 4 x 2 = 8.

We have used this idea, too, as a check for division. How many 23's in 851? Study these operations.

$$\begin{array}{r} 37 \\ 23\overline{)851} \\ \underline{690} \\ 161 \\ \underline{161} \end{array} \qquad\qquad \begin{array}{r} 23 \\ \times\ \underline{37} \\ 161 \\ \underline{690} \\ 851 \end{array}$$

When we ask the question, "What is 851 divided by 23?" we are seeking the answer to the question, "By what number must 23 be multiplied to obtain 851?" In the division and the check above, we see that 37 is the answer to both questions.

If a and b stand for two natural numbers, and a is smaller than b there is a natural number, x, such that

$$a + x = b.$$

The number, x, is the number we find by subtracting a from b. We can explain the meaning of subtraction in terms of the equation $a + x = b$.

In a similar way, if a and b stand for natural numbers, then there may or may not be a natural number, x, such that

$$a \cdot x = b$$

If there is such a natural number, then x is the number we find by dividing b by a. We can explain the meaning of division in terms of the equation $a \cdot x = b$. If $b = 15$ and $a = 3$, then in dividing 15 by 3 we are seeking a number, x, such that:

$$3x = 15$$

Exercises - 6

1. Add the following numbers and check by the inverse operation:

A		B	
(a) $\begin{array}{r}58\\ \underline{41}\end{array}$	(b) $\begin{array}{r}37\\ \underline{2400}\end{array}$	(a) $\begin{array}{r}864\\ \underline{574}\end{array}$	(b) $\begin{array}{r}427\\ \underline{535}\end{array}$
(c) 86 + 27		(c) Eight hundred seventy-six plus four hundred ninety-five is what?	
(d) One hundred six plus eight hundred ninety-seven is what?		(d) What is the sum of 32,098 and 80,605?	
(e) Find the sum of 798 and 508.		(e) Adding 20,009 to 89,991 gives what number?	

2. Subtract the following and check by addition:

(a) 86 - 24	(b) 67 - 28	(a) 916 - 805	(b) 1110 - 1010
(c) 167 - 78		(c) 8991 minus 6989	
(d) If one bookcase will hold 128 books and another 109 books, how many more books does the former hold?		(d) A theatre sold 4789 tickets one month and 6781 tickets the next month. How many more people came to the theatre the second month than came the first month?	
(e) If one building has 900 windows and another 811 windows, how many more windows does the first building contain?		(e) The population of a town was 19,891 people. Five years later the population was 39,110 people. What was the increase of population for the five years?	

3. Multiply the following numbers and check by the inverse operation:

(a) 2 · 241	(b) 734 · 9	(a) 213 · 23	(b) 518 · 76

(c) 20 · 841 (d) 239 · 37

(e) What is the product of 678 and 49?

(c) 509 · 48

(d) What is 87 times itself?

(e) If one truck will carry 2099 boxes, how many boxes will 79 trucks carry?

4. Do the indicated divisions and check by multiplication:

(a) $\frac{96}{3}$ (b) $\frac{936}{9}$ (c) $\frac{168}{12}$

(d) $\frac{357}{21}$

(e) At a party there were 312 pieces of candy. If there were 24 children at the party, how many pieces of candy could each child have?

(a) $\frac{703}{13}$ (b) Divide 20972 by 107

(c) Take 214 into 108712.

(d) How many racks are needed to store 208 chairs, if each rack holds 16 chairs?

(e) A girl scout troop has 49 members. Each member is to sell boxes of cookies. If the troop has 588 boxes to sell, how many boxes will each girl have to sell in order to sell them all?

5. Find a simpler name for:

(a) $39 - (2 + 5)$

(b) $(9 - 3) + 15$

(c) $119 - (20 - 6)$

(d) $(20 + 11) - (6 - 2)$

(e) $(5 \cdot 16) + 8$

(f) $(35 \cdot 42) - 8$

(g) $9 + 201 + (128 \cdot 239)$

(h) $\frac{9120}{95} - 96$

(a) $299 - (97 + 105 + 25)$

(b) $973 + 728 - (728 - 27)$

(c) $(16 \times 24) - 119$

(d) $(18 \times 46) - (17 \times 47)$

(e) $\frac{3307949}{149} - 201$

(f) $(\frac{625}{25}) \times (\frac{50}{25})$

(g) $(19 \cdot 17) + (\frac{5184}{72}) - (35 + 37)$

(h) $\frac{(104 + 21)}{5(2 + 3)}$

(i) $\frac{8199}{9} + \frac{6488}{8}$	(i) $\frac{(610 + 14 + 205) - (4 \cdot 25)}{(3 \cdot 3) \cdot 3}$
(j) $\frac{2460375}{135} + 1$	(j) $35 \times (12 + 7)$

6. Perform the following operations:

(a) Add 16 and 17. From the sum subtract 12.	(a) Find the difference between 47 and 38. Divide this difference by 3 and then add 17.
(b) Subtract 24 from 89. To this difference add 19.	(b) Divide 272 by 16, multiply the quotient by 12 and subtract 100 from the product.
(c) Multiply 27 by 34. Divide the product by 9 and then add 100.	(c) Multiply 12 times 13 and add 39. Divide the sum by the product of 2 and 7.
(d) Find the sum of 9, 9 and 9. From it subtract 4, 6 times.	(d) Add 26 and 42 and divide the sum by 17. To this add 117 and divide this sum by 11.
(e) Take 308, divide it by 28. Multiply the quotient by 5. Subtract 9 from the product.	(e) Find the difference between 87 and 49. Multiply this difference by 10 and subtract 40. Divide this difference by 68 and then add 6.

Betweenness

Earlier we talked about ordering natural numbers. We now locate the sequence of natural numbers as dots.

1 2 3 4 5 6 7 8 9 10
• • • • • • • • • • (How far can we go?)

Now we are ready to ask some questions about numbers between numbers. How many numbers are between 1 and 7? What did you do to find your answer? Let's try these exercises.

Exercises - 7

1. Here is a row of dots running across the page. Beginning at the left end, taking steps one after another in succession, label each dot with a name for a number. Starting, it might look like this:

   . . . .
   one two three four five

   Continue labeling, as far as the edge of the page. Writing the word names makes the picture cumbersome. If we again use the usual numeration, we will have

   . . . .
   1 2 3

   This is called a number scale. Finish labeling the dots of the number scale.

   (a) Write a number scale, using the binary numeration.

   (b) Write a number scale using the seven system.

   In the remaining questions use the decimal numeration:

   (c) What numeral is the label for the dot nearest the right hand margin of the page?

   (d) What numeral is the label for the dot just to the right of the dot labeled '7'?

   (e) What numeral is the label for the dot just to the left of the dot labeled '7'?

   (f) Between the dots labeled '6' and '8' there is only one dot, the dot labeled '7'. How many dots are between the dots labeled '11' and '13'?

   (g) How many dots are between the dots labeled '2' and '4'?

   (h) How many dots are between the dots labeled '5' and '8'?

   We have more numerals which we have not used as labels. What numeral should be the label for the first dot beyond the margin of the paper?

(i) What numeral should be the label for the second dot beyond the right hand margin?

(j) If the label '19' is given to the fifth dot beyond the right hand margin, where does the label '47' appear?

2. (a) How many dots between the dots labeled '1' and '12'?

   (b) How many dots between the dots labeled '12' and '1'?

   (c) How many dots between the dots labeled '51' and '58'?

   (d) How many dots between the dots labeled '27' and '3'?

   (e) How many dots between the dots labeled '84' and '532'?

   (f) How many dots between the dots labeled '38271' and '52964'?

3. At a drill all students line up in a single file and count off in one, two, three fashion. At her turn, Mary says, "eleven." At his turn, Tom says, "seventy-three." How many students are between Mary and Tom?

4. In a list of town voters, arranged alphabetically, Mrs. Beach is listed as number 197 and her sister, Mrs. Warren, is number 15841. How many names are on the register between the names of the sisters?

5. If a number scale is labeled from left to right, the dot labeled '8' will lie on the left of the dot labeled '10'. Since the number 8 is smaller than the number 10, we use the numeral '8' as a label before we use the numeral '10'. We have seen that symbols can be used to say that the number 8 is less than the number 10. We have written "8 < 10". The number scale shows us this sentence when it shows that the dot labeled '8' lies on the left of the dot labeled '10'.

   (a) The dot labeled '12' lies on the left of the dot labeled '17'. Does that mean that 12 < 17?

   (b) How do the dots labeled '482' and '516' lie?

(c) How many dots between the dots labeled '31' and '35'?

(d) What is the label on the dot midway between '31' and '35'?

6. In a stadium, the benches on the same level are labeled with numerals in a number scale. Don is sitting in the seat labeled '24', and Ed is sitting in the seat labeled '37'. Several very fat people want to sit between Don and Ed. Each one of the fat people needs two seats. How many fat people can squeeze in between Don and Ed?

7. There are uniform notches on a shelf. Each one holds a regular size box. An economy size box needs three spaces. Notch labeled '52' and notch labeled '142' are filled, but the shelf between is empty. How many regular size boxes may be put in? How many economy sized boxes?

8. Parking spaces in a factory parking lot are labeled like a number scale. Two cars are parked in the spaces labeled '57' and '80'. The spaces between are empty. Can a fleet of 12 trucks be parked between the two cars, if one truck occupies two spaces?

9. If a and b are numbers, and $a < b$, can it be true that $b < a$? Is it possible that $a = b$? If a, b, and c are numbers, and $a < b$ and $b < c$, then what is the relation between a and c? State your answer in the form:

If $a < b$ and $b < c$, then ?

10. Which is larger, 3 or 7? If you add 2 to each number how do the results compare? Is there a general law? If $a < b$, then what is the relation between $a + c$ and $b + c$? State your answer in the form:

If $a < b$, then ?

11. If a, b, and c are numbers, and b is between a and c, can c be between a and b? Is b between c and a? If b is between a and c, and a is between b and d, where d is a number, what is the relation among b, c, and d?

The Number One

The number 1 is the smallest of the natural numbers. It has several special properties which should be noticed. First, all the natural numbers may be built from 1 by addition; as we have seen: $1 + 1 = 2$, $1 + 2 = 3$, $1 + 3 = 4$, etc. Second the product of any natural number and 1 is that natural number: $1 \cdot 1 = 1$, $1 \cdot 2 = 2$, $1 \cdot 3 = 3$, etc. It is therefore sometimes called the identity element for multiplication, since no number is changed when you multiply it by 1. Also if you divide by 1, the natural number is not altered.

As a matter of fact, if you assume that $1 \cdot 1 = 1$, you can use the distributive property to show that 1 times any natural number is itself. For example, suppose you wished to show that $1 \cdot 5 = 5$. Then you could write:

$$1 \cdot 5 = 1 \cdot (1 + 1 + 1 + 1 + 1) = (1 \cdot 1) + (1 \cdot 1) + (1 \cdot 1) + (1 \cdot 1) +$$
$$(1 \cdot 1) = 1 + 1 + 1 + 1 + 1 = 5$$

Another property that the number 1 has is that $1^{50} = 1$, $1^{158} = 1$, in fact, any power of 1 is 1. Why does this follow from what we had stated above?

The Number Zero

The number zero is even more special than the number 1. It can be used to count in the sense that if you have no apples, you can express that fact by saying that you have zero apples. It is very useful in our notation for numbers since it serves often as a place holder: 11 is quite different from 101 and .0035 is not .35 .

What happens when we multiply, divide, add and subtract with zero? First $3 \cdot 0 = 0$ since this can be interpreted to be three zeroes and if three people have no apples each, they have amongst them no apples.

And $0 \cdot 3$ has a comparable meaning. How many apples are there in "no sets of three apples each"? It must be that $0 \cdot 3 = 0$. We would be disturbed at any other result since we want multiplication to be commutative. Thus zero times any natural number is zero. Also it is natural to define $0 \cdot 0 = 0$. We might say that if we take the set of unicorns and the set of natural numbers between 1 and 2, and write none of them, we obtain a set of no elements.

It is also true that if the product of two whole numbers is zero, one or both of them must be zero. This holds since the product of two natural numbers is a natural number; that is, if neither of them were zero, their product could not be zero.

If we add zero to any natural number we get the number again. If you have no apples and I have three, we have three apples between us. The order in which we add them does not matter. We could express this by:

$$a + 0 = 0 + a = a,$$

for any natural number a. It is true even if a is zero. Hence a could be any whole number. Similarly $a - 0 = a$.

Can we divide zero by any number? What about 0/3? This is the number which when multiplied by 3 gives zero. So 0/3 is zero, which is a number.

Can we divide by zero? We know that 6/2 is 3 because $3 \cdot 2 = 6$. So if 3/0 is a number it should be one which when multiplied by zero gives 3. All the numbers we have had so far give zero when multiplied by zero. It would be very strange to have a number such that when we multiply it by zero we obtain 3. Another difficulty would be that if 3/0 were a number it would be equal to 1/0 since we could divide numerator and denominator by 3. Then 3/0 and 1/0 would be equal and yet when you multiplied the first by 0 you would get 3 and the other by zero you would get 1. For these and other reasons, we exclude division by zero.

We have stated the properties of zero here in terms of natural numbers. But they all hold for other numbers as well.

Exercises - 8

1. If the product of two whole numbers is zero, one of them is zero. Is it true that if the product of two whole numbers is 1, one of them is 1? Is it true that if the product of two whole numbers is 2, one of them is 2? Will the answer be "yes" for any natural number in place of 2?
2. What is the simplest name for the number $3 - 3$? for $n - n$?
3. If $a + c = a$, what must c be?
4. Should we consider 0/0 to be a number? Why or why not?

UNIT IV

FACTORING AND PRIMES

Do you know what the word "factor" means? The idea is familiar even if the word is not. We know that $5 \times 2 = 10$. Or we may write this as "$5 . 2 = 10$." We call the 5 and 2 factors of 10; 6 and 7 are factors of 42, that is $6 . 7 = 42$. Instead of calling one the multiplicand and one the multiplier, we can give them both the same name -- factor. Also, $42 = 2 \times 3 \times 7$ and we call 2, 3, and 7 factors of 42. Really, does it make any difference which is which? From our understanding of the commutative property of multiplication of natural numbers we know that if the order is changed, the product is still the same. Whether it be 5×2 or 2×5, the product is still 10.

What are the factors of 96? Suppose we think one of the factors is 8. Then 96 divided by 8 is 12. We know that two factors of 96 are 8 and 12, for $8 \times 12 = 96$. Are there other factors of 96?

Exercises - 1

A

1. Factor the following:

   (a) 8 (b) 26 (c) 40
   (d) 54 (e) 56 (f) 72
   (g) 7

2. Using the principle that $n . 1 = n$ if n is any number, find simpler names for these numbers.

   (a) $3(5 - 4)$

B

1. Factor the following:

   (a) 16 (b) 28 (c) 144
   (d) 260 (e) 100 (f) 91
   (g) 13

2. Using the principle that $n . 1 = n$ if n is any number, find simpler names for these numbers.

   (a) $8 . \frac{3}{3}$

(b) $(29 - 28) \cdot 5$

(c) $\frac{2}{3} \cdot (56 - 55)$

(d) $57\% \, (75 - 74)$

(e) 7 . -

(f) $\frac{18}{18} \cdot 100\%$

(b) $\frac{5}{6} \cdot \frac{8}{8}$

(c) $3 \cdot (86 - 85)$

(d) $(\frac{5}{5} \cdot 8) \times 9$

(e) $10 \cdot (\frac{4}{4} \cdot 3) \cdot (8 \cdot \frac{9}{9})$

(f) $\frac{50}{51} \cdot (76\frac{1}{2} - 75\frac{1}{2})$

3. For these use the natural numbers from 1 to 30.

   (a) Give the set of numbers that have the factor 1.

   (b) Give the set of numbers that have the factor 2.

   (c) Give the set of numbers that do not have the factor 2.

   (d) Give the set of numbers that may be factored in more than one way. Factor each number in this set as many ways as you can.

   (e) Give the set of numbers that can be factored only one way. Factor each number in this set. What can you say about the factors of a natural number that can only be factored one way?

We have talked about things having something in common. Let's write products of factors of numbers between 1 and 20. The even numbers between 1 and 20 are in Set I. The odd numbers between 1 and 20 are in Set II.

I	II
$1 \times 2 = 2$	$1 \times 3 = 3$
$1 \times 2 \times 2 = 4$	$1 \times 5 = 5$
$1 \times 3 \times 2 = 6$	$1 \times 7 = 7$
$1 \times 4 \times 2 = 8$	$1 \cdot 3 \cdot 3 = 9$
$1 \times 5 \times 2 = 10$	$1 \times 11 = 11$
$1 \times 6 \times 2 = 12$	$1 \times 13 = 13$
$1 \times 7 \times 2 = 14$	$1 \times 3 \times 5 = 15$

1 × 8 × 2 = 16	1 × 17 = 17
1 × 9 × 2 = 18	1 × 19 = 19

Notice that these even numbers all have the factor 2 in common, and none of these odd numbers has the factor 2. Do you suppose that all even numbers have the factor 2? Do you suppose that any odd number has the factor 2?

Exercises - 2

A	B
1. Pick out the even numbers.	1. Pick out the odd numbers.
37 64 31,766 56 101 420,681 102 2568 570,000	50 36,763 $(101)_3$ 738 829,700 $(210)_3$ 4683 10,110 500%
2. Tell whether these numbers are odd or even.	
(a) 2 × 5	(a) 3 × 2 × 6
(b) 3 + 7	(b) 5 + 7 + 1
(c) 6 × 5 × 3	(c) 4 × 7 × 13
(d) 2 + 16	(d) 128 + 36
(e) 7 + 8	(e) 266 + 627
(f) 5 × 13	(f) 3 × 3 × 7
(g) 257 + 361	(g) 25(7 + 9)
(h) 620 + 928	(h) (13 + 26)26
(i) 26 × 58 × 75	(i) (13 × 12) + 76
(j) 33 × 40 × 77	(j) 27 + (5 × 23)
(k) 5271 × 397 × 705	(k) 110 - 66
(l) 1729 + 5285	(l) 115 - 77
3. Perform the following operations:	
Make a counting chart for numbers from 1 through 10 in base two	Make a counting chart for numbers from 1 to 10

numeration. Circle the numerals for even numbers. How can you recognize an even number written in base two numerals. An odd number?

in base two, three, four and five. In which system of numeration are even numbers easily recognized? Do you have any hunches about even numbers written in systems with larger bases?

Prime Numbers

There are some numbers which have as factors only themselves and 1. For instance,

$$2 = 2 \times 1$$
$$3 = 3 \times 1$$
$$5 = 5 \times 1$$
$$7 = 7 \times 1$$
$$11 = 11 \times 1$$
$$13 = 13 \times 1$$
$$17 = 17 \times 1$$
$$19 = 19 \times 1$$

Any natural number which has only two different factors -- itself and 1 -- is called a prime number. Although 1 has as factors only itself and 1, it is not a prime number. The numbers listed above -- 2, 3, 5, 7, 11, 13, 17, and 19, are all prime numbers. Numbers like 4, 6, 9, 12, 15, and 18, are not prime numbers. They are called composite numbers, for they have more than two different factors. For example, the factors of 4 are 1, 2, and 4. What are some other prime numbers? What are some other composite numbers?

Exercises - 3

1. List the numerals for all the prime numbers you can think of between 1 and 100.

2. Do you think you listed every one? You may have missed a few, or perhaps you have included the numerals of some numbers whose factors you did not recognize. In about 200 B. C., there lived a man called Eratosthenes. This man invented a way to find prime numbers smaller than some number you have in mind. In this case the prime numbers are to be less than 100. To use Eratosthenes' method, we proceed as follows:

   (a) Write in order the numerals for the odd natural numbers that are smaller than 100 beginning with 3.

   (b) Starting with "3" cross out every third numeral. Do not cross out "3", but start counting with 5. Thus: 5, 7, ~~9~~, etc.

   (c) Now, starting with "5" cross out every fifth numeral. Include the numerals already crossed out when you count. Some numerals will be crossed out more than once. Do not cross out the "5", but start counting with 7.

   (d) Again, starting with "7", cross out every seventh numeral. Do not cross out the "7" but start counting with 9.

   (e) The next numeral is "9". It is already crossed out. Skip it and go on to "11".

   (f) Continue in this way until you have crossed out all possible numerals. The numerals left, with the numeral

"2", will be the names of the prime numbers less than 100.

Because numerals "drop out" this method is known as "the sieve of Eratosthenes."

Compare this list with your list in question one. Were you 100% correct? Keep this list in your notebook.

(g) Why is the numeral 2 added to the list of prime numbers? Could we have gotten the same list by writing the numerals for all the natural numbers? How would we begin, then? Do you understand our beginning is just a short cut?

(h) Was it necessary to continue to 31, where you would cross out every thirty-first numeral? At what point did you find that you were not crossing out any new numerals? When you moved on to a new step where the starting numeral had not been crossed out, but found that every other numeral in that step had already been crossed out, then you were finished.

3. (a) Using "the sieve of Eratosthenes", find the prime numbers less than 300.

(b) What numeral began the series in which the last numeral was crossed out? (See 2h above.)

(c) How many prime numbers are less than 300?

(d) How many prime numbers are between 1 and 100? between 100 and 200? between 200 and 300?

(e) Separate the natural numbers between 1 and 300 taken in order in groups of 50. In which group are the greatest number of primes? In which group of 50 do the number of primes increase over the previous group of 50?

(f) How many pairs of prime numbers are there such taht their difference is 2? These are called prime twins.

More about Prime Numbers

Let's add some prime numbers --

$$\begin{array}{cccccc} 3 & 3 & 7 & 11 & 11 & 11 \\ \underline{5} & \underline{7} & \underline{5} & \underline{3} & \underline{5} & \underline{7} \\ 8 & 10 & 12 & 14 & 16 & 18 \end{array}$$

and we could keep on writing sums of prime numbers.

Are these sums always even numbers? Is this always true? Remember if you find one example which is not true, then the generalization cannot be made. Also, no matter how many examples we have, unless we have all possible examples, we do not have a proof.

Exercises - 4

1. Find these sums:
   (a) thirty-one plus nineteen
   (b) five plus twenty-nine
   (c) ninety-seven plus one hundred forty-nine
   (d) two hundred seventy-seven plus one hundred sixty-three
   (e) 199 + 233
   (f) 89 + 167
   (g) Do all of these examples ask for the sum of two prime numbers?

(h) Is the sum an even number in each example?

2. Study these sums. Can you make a guess about them?

$4 = 2 + 2$ $26 = 23 + 3$ $100 = 47 + 53$

$6 = 3 + 3$ $28 = 17 + 11$ $102 = 41 + 61$

$8 = 5 + 3$ $30 = 17 + 13$ $104 = 43 + 61$

$10 = 5 + 5$ $32 = 19 + 13$ $106 = 23 + 83$

In 1742, a mathematician named Goldbach made a conjecture, or guess, about these even numbers, in fact, about all even numbers, except the number two, in a letter he wrote to a fellow mathematician, Euler. He guessed that every even number except 2, is the sum of two prime numbers.

(a) Take a few examples and test Goldbach's conjecture. Take some numbers between 1 and 100, others between 100 and 200, others between 200 and 300.

(b) Can you find one even number, other than 2 that is not the sum of two prime numbers?

(c) Can you prove 'Goldbach's conjecture'? Try.

Another Property of Natural Numbers

In finding factors of numbers in Exercises 1, we gave pairs of numbers whose product was the given number. For example, $12 = 3 \times 4$, so two factors are 3 and 4. Another pair is 6 and 2. But the factors of 4 are 2 and 2. So we can say that the factors of 12 which are prime numbers are 3, 2, and 2, or $12 = 3 \times 2 \times 2$. Remember that we do not consider 1 a prime number. And the factors of 6 are 3 and 2. So again, $12 = 3 \times 2 \times 2$ and the factors are 3, 2, and 2. What kind of numbers are 3 and 2? They have what common property?

Let's study another example. If we can find factors of factors, we will do it. What are the factors of 24?

$24 = 12 \times 2$

$24 = (3 \times 4) \times 2$

$24 = (3 \times 2 \times 2) \times 2$

$24 = 3 \times 2 \times 2 \times 2$

$24 = 4 \times 6$

$24 = (2 \times 2) \times (2 \times 3)$

$24 = 2 \times 2 \times 2 \times 3$

$24 = 8 \times 3$

$24 = (4 \times 2) \times 3$

$24 = (2 \times 2 \times 2) \times 3$

$24 = 2 \times 2 \times 2 \times 3$

What do you observe? The prime factors of 24 are 2, 2, 2, and 3. One set is not in that order. Does that make any difference?

In finding factors of a number, we will find prime numbers. Every composite number can be factored into primes in only one way, except for order. This is called the unique factorization property of natural numbers.

Exercises - 5

1. Factor completely. (That is, find the prime factors.) Express each number as the product of its prime factors.

	A		B
(a)	6	(a)	10
(b)	9	(b)	25
(c)	12	(c)	18
(d)	30	(d)	154
(e)	35	(e)	45
(f)	37	(f)	47
(g)	100	(g)	100
(h)	74	(h)	315
(i)	105	(i)	231
(j)	42	(j)	108
(k)	79	(k)	91

(l) 345	(l) 128
(m) 300	(m) 729
(n) 72	(n) 1000
(o) 64	(o) 5280

2. Examine the products in question 1. If any products use the same factor more than once, rewrite that product, taking advantage of the exponent notation.

3. Factor the numbers listed here in as many ways as possible using only two factors each time. Because of the commutative property, we shall say 3 × 5 is not differnet from 5 × 3.

A	B
(a) 6	(a) 10
(b) 8	(b) 16
(c) 24	(c) 72
(d) 100	(d) 81
(e) 150	(e) 216

4. Study 30 = 2 × 3 × 5. How should this set of factors be grouped to show that 30 = 2 × 15? to show that 30 = 6 × 5?

5. (a) Factor 770 completely. Group the factors to show all the possible products that will equal 770.

4. Why should a number such as 78 have more possible different pairs of factors than 77? or why should 210 have more possible different pairs of factors than 254?

5. (a) If a number is the product of two different prime numbers, in how many different

(b) Factor each of the following numbers completely. Group the factors in each case to show all the possible ways the number is the product of two natural numbers.

(1) 42
(2) 66
(3) 78
(4) 12
(5) 18
(6) 48
(7) 49
(8) 75
(9) 64

ways may the number be factored using different pairs of factors?

(b) If a number is the product of three different prime numbers, in how many different ways may the number be factored using different pairs of factors?

(c) Fill in this chart.

Number	Complete Factorization	No. of Complete Factors	Different Ways to Factor Using Pairs	No. of Ways to Factor Using Pairs
55				
130	2x5x13	3	1x130, 2x65 10x13, 5x26	4
770				
2310				
28,014				

Is there any pattern in the number of ways such numbers may be

(c) Study

114 = 2 . 3 . 19

50 = 2 . 5 . 5

Why are there more possible ways to obtain 114 as a product of natural numbers than to obtain 50? Each of the numbers has 3 factors.

6. If I have 112 tulip bulbs to plant and would like to plant them to make a series of equal rows, what possible arrangements could I use?

factored using pairs of factors?

6. If I have 1000 chairs to set up in an orderly fashion in a large auditorium, and want to make a series of equal rows, what possible arrangements could I make? If I would like the number of rows to be as close as possible to the number of chairs in each row, which possibility should I choose?

Greatest Common Factor (g.c.f.)

We have been looking for common properties in sets of things, that is, we have been finding something which each member of the set has. In our study of even numbers we saw that each even number has a factor of 2. So, we said even numbers have a common factor, 2. Let's find common factors for other sets of numbers. Is there a common factor for 10 and 15?

Factors of 10: 5 and 2

Factors of 15: 5 and 3

They have a common factor, 5.

Is there a common factor for 24 and 36?

Express 24 as a product of its prime factors. $24 = 2 \times 2 \times 2 \times 3$

Express 36 as a product of its prime factors. $36 = 2 \times 2 \times 3 \times 3$

Yes, they have common factors of 2, 2 and 3. We can say then that their largest factor in common, or their greatest common factor, is $2 \times 2 \times 3$ or 12.

We can use factoring to help us change from one name for a fraction to another. For example, we know that

$$\frac{2}{4} = \frac{1}{2}$$

But let's use some of the things we have learned about greatest common factors. We write the factors of the numerator and denominator.

$$\frac{2}{4} = \frac{1 \cdot 2}{2 \cdot 2} = \frac{1}{2} \cdot \frac{2}{2}$$

But another name for $\frac{2}{2}$ is 1. So we can write

$$\frac{2}{4} = \frac{1}{2} \cdot 1$$

And we know that any number times 1 is itself. Then, $\frac{2}{4} = \frac{1}{2}$.

Study these:

(a) $\frac{18}{24} = \frac{6 \cdot 3}{6 \cdot 4} = \frac{6}{6} \cdot \frac{3}{4} = 1 \cdot \frac{3}{4} \quad \frac{18}{24} = \frac{3}{4}$ (g.c.f. is 6)

(b) $\frac{28}{36} = \frac{4 \cdot 7}{4 \cdot 9} = \frac{4}{4} \cdot \frac{7}{9} = 1 \cdot \frac{7}{9}$ $\frac{28}{36} = \frac{7}{9}$ (g.c.f. is 4)

(c) $\frac{36}{84} = \frac{3 \cdot 3 \cdot 2 \cdot 2}{7 \cdot 3 \cdot 2 \cdot 2} = \frac{3}{7} \cdot \frac{(3 \cdot 2 \cdot 2)}{(3 \cdot 2 \cdot 2)} = \frac{3}{7} \cdot 1$ $\frac{36}{84} = \frac{3}{7}$ (g.c.f. is 12)

Sometimes we find it difficult to recognize the greatest common factor for two or more numbers. We may use prime factors, just as we did in the last example above, to help us.

Exercises - 6

1. Factor each number completely and find the g.c.f.

   (a) 35, 21 and 49 (b) 21, 27 and 15 (c) 42, 147 and 105
   (d) 60, 42 and 66 (e) 24, 60 and 84 (f) 78, 13 and 39
   (g) 28, 56 and 14

2. Simplify, that is, carry out the indicated operations.

   (a) $\frac{5 \cdot 3}{7 \cdot 3}$ (b) $\frac{3 \cdot 5}{7 \cdot 3}$ (c) $\frac{8 \cdot 2}{3 \cdot 8}$
   (d) $\frac{13 \cdot 4}{13 \cdot 5}$ (e) $\frac{2 \cdot 5 \cdot 3}{11 \cdot 5 \cdot 3}$ (f) $\frac{2 \cdot 5 \cdot 3}{2 \cdot 3 \cdot 13}$
   (g) $\frac{(2 \cdot 17) \cdot 11}{5 \cdot (2 \cdot 17)}$ (h) $\frac{(5 \cdot 3) \cdot 6}{7 \cdot (3 \cdot 5)}$ (i) $\frac{(2 \cdot 2 \cdot 5) \cdot 3}{(2 \cdot 2 \cdot 5) \cdot 7}$
   (j) $\frac{5 \cdot (3 \cdot 7 \cdot 3)}{(3 \cdot 3 \cdot 7) \cdot 11}$ (k) $\frac{2 \cdot 15}{21 \cdot 7}$ (l) $\frac{3 \cdot 5}{13 \cdot 2}$
   (m) $\frac{2 \times 2 \times 7 \times 3}{3 \times 2 \times 2 \times 2}$

3. Find simpler names for these numbers.

   (a) $\frac{10}{35}$ (b) $\frac{42}{60}$ (c) $\frac{49}{56}$ (d) $\frac{100}{166}$ (e) $\frac{36}{63}$ (f) $\frac{84}{112}$
   (g) $\frac{100}{164}$ (h) $\frac{336}{633}$ (i) $\frac{5280}{5760}$ (j) $\frac{1456}{2184}$ (k) $\frac{945}{1080}$
   (l) $\frac{8772}{20,468}$

Multiples of Numbers

In learning the multiplication facts, we learned multiples of numbers. For example, multiples of 4 between 1 and 40 are 4, 8, 12, 16, 20, 24, 28, 32, and 36. A multiple of 4 is a number that has 4 as a factor. Study the multiplication table again. What are multiples of other numbers? What numbers are multiples for different numbers? What numbers have the same multiple? Do you see any patterns in the multiples?

x	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9
2	0	2	4	6	8	10	12	14	16	18
3	0	3	6	9	12	15	18	21	24	27
4	0	4	8	12	16	20	24	28	32	36
5	0	5	10	15	20	25	30	35	40	45
6	0	6	12	18	24	30	36	42	48	54
7	0	7	14	21	28	35	42	49	56	63
8	0	8	16	24	32	40	48	56	64	72
9	0	9	18	27	36	45	54	63	72	81

Exercises - 7

1. What multiples of 6 are less than 100?
2. What multiples of 14 are less than 100?
3. What multiples of 9 are between 250 and 300?
4. What multiples of 23 are between 300 and 350?
5. For what natural numbers less than 10 do all multiples have decimal numerals which end in 0, 2, 4, 6, 8?
6. For what natural numbers less than 10 are multiples even numbers?
7. For what natural numbers less than 20 do all multiples have decimal numerals which end in 0 or 5?
8. For what natural numbers less than 20 can we find nine

multiples whose decimal numerals end in 1, 2, 3, 4, 5, 6, 7, 8, 9 respectively?

9. What number does not have itself as a multiple?
10. What natural number less than 20 has multiples that are only odd numbers?
11. Can a natural number that is a composite number have a prime number as a multiple?
12. Write the names of six multiples of 12 using duodecimal numeration.
13. Write the names of six multiples of 7 using duodecimal numeration.
14. Write the names of six multiples of 2 using binary notation.
15. Outside white paint comes only in gallon cans. How many cans must be bought if 35 quarts are needed?
16. For refreshments at a campfire, each member is to receive 3 marshmallows. Marshmallows come in packages of 16, costing 13 cents a package. If 15 people are at the campfire, how many packages are needed?
17. If auditorium chairs come in sections containing 6 seats, how many sections will be needed for an audience of 100? of 150? of 200? of 201? of 202? of 203?

Least Common Multiple (l.c.m.)

We have learned that the greatest common factor for two or more numbers is the largest factor common to those numbers. The greatest common factor of 4, 6 and 8, is 2. It is the largest factor that is common to each.

We also know that:

Multiples of 4 are 4, 8, 12, 16, 20, 24, 28, 32, 36, 40, 44, 48, etc.
Multiples of 6 are 6, 12, 18, 24, 30, 36, 42, 48, 54, 60, etc.
Multiples of 8 are 8, 16, 24, 32, 40, 48, 56, 64, 72, 80, etc.
What numbers are multiples of all three? Which is the smallest one? We call the smallest common multiple for two or more numbers their *least common multiple*.

We make use of this idea in finding like denominators in adding and subtracting fractions. It is true that we can use any common multiple. If we found the product of 4, 6, and 8, or 4 × 6 × 8, we would have another multiple, 192. However, it is easier to add fractions if we can find the smallest common multiple. Factoring helps us in finding it.

Find the least common multiple of 4, 6 and 8.

$4 = 2 \times 2$

$6 = 2 \times 3$

$8 = 2 \times 2 \times 2$

The least common multiple of these numbers must have all the different prime factors. Each of these prime factors will appear as many times as it occurs in the number where it appears most frequently.

Least common multiple (l.c.m.) for 4, 6 and 8 is $2 \times 2 \times 2 \times 3$ or 24.

Study these examples:

(a) $\frac{2}{3} + \frac{5}{9}$ 27 is *a* common multiple of 3 and of 9.

$$\frac{2}{3} \cdot \frac{9}{9} + \frac{5}{9} \cdot \frac{3}{3} = \frac{18}{27} + \frac{15}{27} = \frac{33}{27} = 1 + \frac{6}{27} = 1 + \frac{2}{9}$$

By finding the least common multiple,

$$\frac{2}{3} \cdot \frac{3}{3} + \frac{5}{9} = \frac{6}{9} + \frac{5}{9} = \frac{11}{9} = 1 + \frac{2}{9}$$

(b) $\frac{3}{15} + \frac{5}{18} + \frac{7}{12}$ Finding the least common multiple

$$\frac{3}{15} \cdot \frac{12}{12} + \frac{5}{18} \cdot \frac{10}{10} + \frac{7}{12} \cdot \frac{15}{15} =$$

$$\frac{36}{180} + \frac{50}{180} + \frac{105}{180} = \frac{191}{180} = 1 + \frac{11}{180}$$

15 = 5 × 3
18 = 3 × 3 × 2
12 = 2 × 2 × 3
l.c.m.: 5 × 3 × 3 × 2 × 2 or 180

(c)

$$7\tfrac{2}{9} = 7\tfrac{2}{9} \cdot \frac{2}{2} = 7\tfrac{4}{18}$$
$$5\tfrac{1}{3} = 5\tfrac{1}{3} \cdot \frac{6}{6} = 5\tfrac{6}{18}$$
$$2\tfrac{1}{2} = 2\tfrac{1}{2} \cdot \frac{9}{9} = 2\tfrac{9}{18}$$

$$14\tfrac{19}{18} = 15 + \frac{1}{18} \text{ or } 15\tfrac{1}{18}$$

Finding l.c.m.,
9 = 3 × 3
3 = 3 × 1
2 = 2 × 1
l.c.m.: 3 × 3 × 2 = 18

Exercises - 8

1. Find the multiples of the following numbers which are less than 100.

A	B
(a) 2, 3 and 4	(a) 2, 4 and 8
(b) 3, 6 and 9	(b) 6, 7 and 8
(c) 7, 8 and 9	(c) 11 and 5
(d) 13 and 3	(d) 14 and 12

2. Find the common multiples of the numbers listed in each part of Problem 1.
3. Find the least common multiple of the numbers listed in each part of Problem 1.
4. Find the least common multiple of:

   (a) 6, 10 and 14

(a) 6, 10 and 14

(b) 20, 22 and 12

(c) 70, 21 and 30

(d) 5, 20 and 16

(e) 9, 36 and 18

(f) 5, 6 and 7

(a) 9, 15 and 21

(b) 12, 14 and 16

(c) 13, 15 and 17

(d) 20, 40 and 50

(e) 26, 12 and 39

(f) 10, 75 and 45

5. (a) Find all the common multiples of 3, 4, and 8 which are less than 75.

(b) Which is the least common multiple?

(c) Which multiple is next greater than the l.c.m.?

(d) Whcih multiple is next greater than the last one?

(e) Do you have a hunch what the next two greater multiples will be?

(a) What is the least common multiple of 5, 4 and 16?

(b) What is another common multiple of 5, 4 and 16?

(c) What common multiple is between 200 and 250?

(d) What common multiple is between 550 and 600?

(e) What hunch do you have about common multiples when compared with the least common multiple?

(f) What is the greatest common multiple of 5, 4 and 16?

6. Perform the indicated operation.

(a) $\frac{2}{3} + \frac{7}{9}$

(b) $\frac{3}{14} + \frac{9}{16}$

(a) $\frac{5}{20} + \frac{3}{16}$

(b) $\frac{13}{24} - \frac{5}{18}$

(c) $\frac{7}{8} - \frac{3}{5}$

(d) $\frac{13}{16} - \frac{3}{20}$

(e) $\frac{9}{24} + \frac{9}{16}$

(f) $\frac{3}{4} + \frac{2}{5} + \frac{5}{6}$

(g) $\frac{5}{8} + \frac{7}{12} + \frac{1}{16}$

(h) $\frac{4}{28} - \frac{1}{7}$

(i) $\frac{1}{10} + \frac{15}{16} - \frac{3}{20}$

(j) $(\frac{13}{25} - \frac{1}{10}) - \frac{1}{15}$

(k) $\frac{3}{25} - (\frac{1}{10} - \frac{1}{15})$

(c) $\frac{2}{3} + \frac{3}{4} + \frac{4}{5}$

(d) $\frac{7}{8} + \frac{1}{14} - \frac{2}{35}$

(e) $(\frac{17}{18} - \frac{1}{9}) - \frac{2}{5}$

(f) $\frac{2}{3} - (\frac{4}{13} - \frac{11}{39})$

(g) $\frac{43}{50} - (\frac{2}{55} + \frac{19}{60})$

(h) $\frac{9}{14} - 6\frac{1}{6}$

(i) $\frac{13}{22} + (\frac{37}{55} - \frac{10}{33})$

(j) $(\frac{33}{56} + \frac{33}{42}) - \frac{35}{96}$

(k) $(\frac{8}{15} + \frac{11}{18}) - (\frac{4}{9} + \frac{3}{10})$

UNIT IV-A

SUPPLEMENTARY TESTS FOR DIVISIBILITY AND REPEATING DECIMALS

1. Introduction: This monograph is for the student who has studied a little about repeating decimals, numeration systems in different bases, and tests for divisibility (casting out the nines, for instance) and would like to carry his investigation a little further, under guidance. The purpose of this monograph is to give this guidance; it is not just to be read. You will get the most benefit from this material if you will first read only up to the first set of exercises and then without reading any further do the exercises. They are not just applications of what you have read, but to guide you in discovery of further important and interesting facts. Some of the exercises may suggest other questions to you. When this happens, see what you can do toward answering them on your own. Then, after you have done all that you can do with that set of exercises, go on to the next section. There you will find the answers to some of your questions, perhaps, and a little more information to guide you toward the next set of exercises.

The most interesting and useful phase of mathematics is the discovery of new things in the subject. Not only is this the most interesting part of it, but this is a way to train yourself to discover more and more important things as time goes on. When you learned to walk, you needed a helping hand, but you really had not learned until you could stand alone. Walking was not new to mankind -- lots of people had walked before -- but it was new to you. And whether or not you would eventually discover places in your walking which no man had ever seen before, was unimportant. It was a great thrill when you first found that you could walk, even though it looked like a stagger to other people. So, try learning to walk in mathematics. And be independent -- do not accept any more help than is necessary.

2. Casting out the nines. You may know a very simple and interesting way to tell whether a number is divisible by 9. It is based on the fact that a number is divisible by 9 if the sum of its digits is divisible by 9 and if the sum of its digits is divisible by 9, the number is divisible by 9. For instance, consider the number 156782. The sum of its digits is 1 + 5 + 6 + 7 + 8 + 2 which is 29. But 29 is not divisible by 9 and hence the number 156782 is not divisible by 9. If the second digit had been a 3 instead of 5, or if the last digit had been 0 instead of 2, the number would have been divisible by 9 since the sum of the digits would have been 27 which is divisible by 9. The test is a good one because it is easier to add the digits than to divide by 9. Actually we could have been lazy and instead of dividing 29 by 9, use the fact again, add 2 and 9 to get 11, add the 1 and 1 to get 2 and see that since 2 is not divisible by 9, then the original six digit number is not divisible by 9.

Why is this true? Merely dividing the given number by 9 would have tested the result but from that we would have no idea why it would hold for any other number. We can show what is happening by writing out the number 156,782 according to what it means in the decimal notation:

$$1 \times 10^5 + 5 \times 10^4 + 6 \times 10^3 + 7 \times 10^2 + 8 \times 10 + 2 =$$

$$1 \times (99999 + 1) + 5 \times (9999 + 1) + 6 \times (999 + 1) + 7 \times (99 + 1) +$$

$$8 \times (9 + 1) + 2.$$

Now by the distributive property, $5 \times (9999 + 1) = 5 \times 9999 + 5 \times 1$ and similarly for the other expressions. Also we may rearrange the numbers in the sum since addition is commutative. So our number 156,782 may be written

$$1 \times (99999) + 5 \times (9999) + 6 \times (999) + 7 \times (99) + 8 \times 9 +$$

$$(1+5+6+7+8+2).$$

Now 99999, 9999, 999, 99, 9 are all divisible by 9, the products involving these numbers are divisible by 9 and the sum of these products is divisible by 9. Hence the original number will be divisible by 9 if (1+5+6+7+8+2) is divisible by 9. This sum is the sum of the digits of the given number. Writing it out this way shows that no matter what the given number is, the same principle holds.

Exercises

1. Choose four numbers and by the above method test whether or not they are divisible by 9. When they are not divisible by 9, compare the remainder when the sum of the digits is divided by 9 with the remainder when the number is divided by 9. Could you guess some general fact from this? If you can, test it with a few other examples.
2. Given two numbers. First, add them, divide by 9 and take the remainder. Second, find the sum of their remainders after each is divided by 9, divide the sum by 9 and take the remainder. The final remainders in the two cases are the same. For instance, let the numbers be 69 and 79. First, their sum is 148 and the remainder when 148 is divided by 9 is 4. Second, the remainder when 69 is divided by 9 is 6 and when 79 is divided by 9 is 7; the sum of 6 and 7 is 13, and if 13 is divided by 9, the remainder is 4. The result is 4 in both cases. Why are the two results the same no matter what numbers are used instead of 69 and 79? Would a similar result hold for a sum of three numbers?
   (Hint: write 69 as 7 x 9 + 6)
3. If in the previous exercise we divided by 7 instead of 9, would the remainders by the two methods be the same? Why or why not?

58

4. Suppose in exercise 2 we considered the product of two numbers instead of their sum. Would the corresponding result hold? That is, would the remainder when the product of 69 and 79 is divided by 9 be the same as when the product of their remainders is divided by 9? Would this be true in general? Could they be divided by 23 instead of 9 to give a similar result? Could similar statements be made about products of more than two numbers?

5. Use the result of the previous exercise to show that 10^{20} has a remainder of 1 when divided by 9. What would its remainder be when it is divided by 3? By 99?

6. What is the remainder when 7^{20} is divided by 6?

7. You know that when a number is written in the decimal notation, it is divisible by 2 if its last digit is divisible by 2, and divisible by 5 if its last digit is 0 or 5. Can you devise a similar test for divisibility by 4, 8, or 25?

8. In the following statement, fill in both blanks with the same number so that the statement is true:

   A number written in the system to the base twelve is divisible by _____ if its last digit is divisible by _____. If there is more than one answer, give the others, too. If the base were seven instead of twelve, how could the blanks be filled in? (Hint: one answer for base twelve is 6)

9. One could have something like "decimal" equivalents of numbers in numeration systems to bases other than ten. For instance, in the numeration system to the base seven, the septimal equivalent of $5(1/7) + 6(1/7)^2$ would be written .56 just as the decimal equivalent of $5(1/10) + 6(1/10)^2$ would be written .56 in the decimal system. The number .142857142857... is equal to 1/7 in the

decimal system and in the system to the base seven would be written .1 . On the other hand, $.1 = (.04620462...)_7$. What numbers would have terminating septimals in the numeration system to the base 7? What would the septimal equivalent of 1/5 be in the system to the base 7? (Hint: remember that if the only prime factors of a number are 2 and 5, the decimal equivalent of its reciprocal terminates)

10. Use the result of exercise 3 to find the remainder when 9 + 16 + 23 + 30 + 37 is divided by 7. Check your result by computing the sum and dividing by 7.
11. Use the results of the previous exercises to show that $10^{20} - 1$ is divisible by 9, $7^{108} - 1$ is divisible by 6.
12. Using the results of some of the previous exercises if you wish, shorten the method of showing that a number is divisible by 9 if the sum of its digits is divisible by 9.
13. See page 15.

3. Why does casting out the nines work? First let us review some of the important results shown in the exercises which you did above. In exercises 2, you showed that to get the remainder of the sum of two numbers, after division by 9, you can divide the sum of their remainders by 9 and find its remainder. Perhaps you did it this way (there is more than one way to do it; yours may have been better). You know in the first place that any natural number may be divided by 9 to get a quotient and remainder. For instance, if the number is 725, the quotient is 80 and the remainder is 5. Furthermore 725 = 80 x 9 + 5 and you could see from the way this is written that 5 is the remainder. Thus, using the numbers in the exercise, you would write 69 = 7x9 + 6 and 79 = 8x9 + 7. Then 69 + 79 = 7 x 9 + 6 + 8 x 9 + 7. Since the sum of two numbers is

commutative, you may reorder the terms and have $69 + 79 = 7 \times 9 + 8 \times 9 + 6 + 7$. Then, by the distributive property, $69 + 79 = (7+8) \times 9 + 6 + 7$. Now the remainder when $6 + 7$ is divided by 9 is 4 and $6 + 7$ can be written $1 \times 9 + 4$. Thus $69 + 79 = (7 + 8 + 1) \times 9 + 4$. So, from the form it is written in, we see that 4 is the remainder when the sum is divided by 9. It is also the remainder when the sum of the remainders, $6 + 7$, is divided by 9.

Writing it out in this fashion is more work than making the computations the short way but it does show what is going on and why similar results would hold if 69 and 79 were replaced by any other numbers, and, in fact, we could replace 9 by any other number as well. One way to do this is to use letters in place of the numbers. This has two advantages. In the first place it helps us be sure that we did not make use of the special properties of the numbers we had without meaning to do so. Secondly, we can, after doing it for letters, see that we may replace the letters by any numbers. So, in place of 69 we write the letter *a*, and in place of 79, the letter *b*. When we divide the number *a* by 9 we would have a quotient and a remainder. We can call the quotient the letter q and the remainder, the letter r. Then we would have

$$a = (q \times 9) + r$$

where r is zero or some natural number less than 9. We could do the same for the number b, but we should not let q be the quotient since it might be different from the quotient when *a* is divided by 9. We here could call the quotient q' and the remainder r'. Then we would have

$$b = (q' \times 9) + r'.$$

Then the sum of a and b will be

$$a + b = (q \times 9) + r + (q' \times 9) + r'.$$

We can use the commutative property to have

$$a + b = (q \times 9) + (q' \times 9) + r + r'$$

and the distributive property to have

$$a + b = (q + q') \times 9 + r + r'.$$

Then if $r + r'$ were divided by 9, we would have a quotient which we might call q'' and a remainder r''. Then $r + r' = (q'' \times 9) + r''$ and

$$\begin{aligned} a + b &= (q + q') \times 9 + (q'' \times 9) + r'' \\ &= (q + q' + q'') \times 9 + r''. \end{aligned}$$

Now r'' is zero or less than 9 and hence it is not only the remainder when $r + r'$ is divided by 9 but also the remainder when $a + b$ is divided by 9. So as far as the remainder goes, it does not matter whether you add the numbers or add the remainders and divide by 9.

The solution of exercise 4 goes the same way as that for exercise 2 except that we multiply the numbers. Then we would have

$$\begin{aligned} 69 \times 79 &= (7 \times 9 + 6) \times (8 \times 9 + 7) \\ &= 7 \times 9 \times (8 \times 9 + 7) + 6 \times (8 \times 9 + 7) \\ &= 7 \times 9 \times 8 \times 9 + 7 \times 9 \times 7 + 6 \times 8 \times 9 + 6 \times 7 \end{aligned}$$

The first three products are divisible by 9 and by what we showed in exercise 2, the remainder when 69×79 is divided by 9 is the same as the remainder when $0 + 0 + 0 + 6 \times 7$ is divided by 9. So in finding the remainder when a product is divided by 9 it makes no difference whether we use the product or the product of the remainders.

If we were to write this out in letters as we did the sum, it would look like this:

$$\begin{aligned} a \times b &= (q \times 9 + r) \times (q' \times 9 + r') \\ &= q \times 9 \times q' \times 9 + q \times 9 \times r' + r \times q' \times 9 + r \times r' \end{aligned}$$

Again each of the first three products is divisible by 9 and hence the remainder when a x b is divided by 9 is the same as when r x r' is divided by 9.

We used the number 9 all the way above, but the same conclusions would follow just as easily for any number in place of 9, such as 7, 23, etc. We could have used a letter for 9 also but this seems like carrying it too far.

There is a shorter way of writing some of the things we had above. When letters are used, we usually omit the multiplication sign and write ab instead of a x b and 9q in place of 9 x q. Hence the last equation above could be abbreviated to

$$ab = qq'9 \times 9 + qr'9 + rq'9 + rr'$$

or

$$ab = 9 \times 9qq' + 9qr' + 9rq' + rr'.$$

But this is not especially important right now.

So let us summarize our results so far: The remainder when the sum of two numbers is divided by 9 (or any other number) is the same as the remainder when the sum of the remainders is divided by 9 (or the same other number). The same procedure holds for the product in place of the sum.

These facts may be used to give quite a short proof of the important result stated in exercise 13. Consider again the number 156,782. This is written in the usual form:

$$1 \times 10^5 + 5 \times 10^4 + 6 \times 10^3 + 7 \times 10^2 + 8 \times 10 + 2.$$

Now the result stated above for the product, the remainder when 10^2 is divided by 9 is the same as when the product of the remainders 1 x 1 is divided by 9, that is, the remainder is 1. Similarly 10^3 has a remainder 1 x 1 x 1 when divided by 9 and hence 1. So all the powers of ten have a remainder 1 when divided by 9. Thus, by the result stated above for the sum, the remainder when 156,782 is divided by 9 is the same as the remainder when $1 \times 1 + 5 \times 1 + 6 \times 1 + 7 \times 1 + 8 \times 1 + 2$ is divided by 9. This last is just the sum of the digits. Writing it this way it is easy to see that this works for any number.

Now we can use the result of exercise 13 to describe a check called "casting out the nines" which is not used much in these days of computing machines, but which is still interesting. Consider the product 867 x 935. We indicate the following calculations:

867 sum of digits: 21 sum of digits 3
934 sum of digits: 16 sum of digits 7

Product 809,778 Product: 3 x 7 = 21
Sum of digits 8 + 0 + 9 + 7 + 7 + 8 = 39
Sum of digits 3 + 9 = 12 Sum of digits: 2 + 1 = 3
Sum of digits 1 + 2 = 3

Since the two results 3 are the same, we have at least some check on the accuracy of the results.

Exercises

1. Try the method of checking for another product. Would it also work for a sum? If so try it also.
2. Explain why this should come out as it does.
3. If a computation checks this way, show that it still could be wrong. That is, in the example given above, what would be an incorrect product that would still check?
4. Given the number $5 \cdot 7^5 + 3 \cdot 7^4 + 2 \cdot 7^3 + 1 \cdot 7^2 + 4 \cdot 7 + 3$. What is its remainder when it is divided by 7? What is its remainder when it is divided by 6? by 3?
5. Can you find any short-cuts in the example above analogous to casting out the nines?
6. In a numeration system to the base 7 what would be the result corresponding to that in the decimal system which gives casting out the nines?

7. The following is a trick based on casting out the nines. Can you see how it works? You ask someone to pick a number -- it might be 1678. Then you ask him to form another number from the same digits in a different order -- he might take 6187. Then you ask him to subtract the smaller from the larger and give you the sum of all but one of the digits in the result. (He would have 4509 and might add the last three to give you 14). All of this would be done without your seeing any of the figuring. Then you would tell him that the other digit in the result is 4. Does the trick always work?

One method of shortening the computation for a test by casting out the nines, is to discard any partial sums which are 9 or a multiple of 9. For instance, in the example given, we did not need to add all the digits in 810,645. We could notice that 8 + 1 = 9 and 4 + 5 = 9 and hence the remainder when the sum of the digits is divided by 9 would be 0 + 6, which is 6. Are there other places in the check where work could have been shortened? We thus, in a way, throw away the nines. It was from this that the name "casting out the nines" came.

By just the same principle, in a number system to the base 7 one would cast out the sixes, to the base 12 cast out the elevens, etc.

4. Divisibility by 11. There is a test for divisibility by 11 which is not quite so simple as that for divisibility by 9 but is quite easy to apply. In fact, there are two tests. We shall start you on one and let you discover the other for yourself. Suppose we wish to test the number 17945 for divisibility by 11. Then we can

write it as before

$$1 \cdot 10^4 + 7 \cdot 10^3 + 9 \cdot 10^2 + 4 \cdot 10^1 + 5.$$

The remainders when 10^2 and 10^4 are divided by 11 are 1. But the remainders when $10, 10^3, 10^5$ are divided by 11 are 10. Now 10 is equal to 11 - 1. $10^3 = 10^2(11 - 1)$, $10^5 = 10^4(11 - 1)$. That is enough. Perhaps we have told you too much already. It is your turn to carry the ball.

Exercises

1. Without considering 10 to be 11-1, can you from the above devise a test for divisibility by 11?
2. Noticing that 10 = 11 - 1 and so forth as above, can you devise another test for divisibility by 11?

We hope you were able to devise the two tests suggested in the previous exercises. For the first, we could group the digits and write the number 17945 as $1 \times 10^4 + 79 \times 10^2 + 45$. Hence the remainder when the number 17945 is divided by 11 should be the same as the remainder when 1 + 79 + 45 is divided by 11, that is 1 + 2 + 1 = 4. (2 is the remainder when 79 is divided by 11, etc.) This method would hold for any number.

The second method requires a little knowledge of negative numbers (either review them or, if you have not had them, omit this paragraph). We could consider -1 as the remainder when 10 is divided by 11. Then the original number would have the same remainder as the remainder when $1 + 7(-1)^3 + 9 + 4(-1) + 5$ is divided by 11, that is, when $5 - 4 + 9 - 7 + 1$ is divided by 11. This last sum is equal to 4 which was what we got the other way. By this test we start at the right and alternately add and subtract digits. This is simpler than the other one.

Exercises

1. Test several numbers for divisibility by 11 using the two methods described above. Where the numbers are not divisible, find the remainders by the method given.
2. In a number system to the base 7, what number could we tes for divisibility in the same way that we tested for 11 in the decimal system? Would both methods given above work for base 7 as well?
3. To test for divisibility by 11 we grouped the digits in pairs. What number or numbers could we test for divisibility by grouping the digits in triples? For example we might consider the number 157892. We could form the sum of 157 and 892. For what numbers would the remainders be the same?
4. Answer the questions raised in exercise 3 in a numeral system to base 7 as well as in a numeral system to base 12.
5. In the repeating decimal for 1/9 in the decimal system there is one digit in the repeating portion; in the repeating decimal for 1/11 in the decimal system, there are two digits in the repeating portion. Is there any connection between these facts and the tests for divisibility for 9 and 11? What would be the connection between repeating decimals and the questions raised in exercise 3 above?
6. Could one have a check in which 11's were "cast out"?
7. Can you find a trick for 11 similar to that in exercise 1 above?

5. Divisibility by 7. There is not a very good test for divisibility by 7 in the decimal system. (In a numeration system to what base would there be a good test?) But it is worth looking into since we can see the connection between tests for divisibility and the

repeating decimals. Consider the remainders when the powers of 10 are divided by 7. We put them in a little table:

n	1	2	3	4	5	6	7
Remainder when 10^n is divided by 7	3	2	6	4	5	1	3

If you compute the decimal equivalent for 1/7 you will see that the remainders are exactly the numbers in the second line of the table in the order given. Why is this so? This means that if we wanted to find the remainder when 7984532 is divided by 7 we would write

$$7 \times 10^6 + 9 \times 10^5 + 8 \times 10^4 + 4 \times 10^3 + 5 \times 10^2$$
$$+ 3 \times 10 + 2$$

and replace the various powers of 10 by their remainders in the table to get

$$7 \times 1 + 9 \times 5 + 8 \times 4 + 4 \times 6 + 5 \times 2 + 3 \times 3 + 2$$

We would have to compute this, divide by 7 and find the remainder. That would be as much work as dividing by 7 in the first place. So this is not a practical test but it does show the relationship between the repeating decimal and the test.

Notice that the sixth power of 10 has a remainder of 1 when it is divided by 7. If instead of 7 some other number is taken which has neither 2 nor 5 as a factor, 1 will be the remainder when some power of 10 is divided by that number. For instance, there is some power of 10 which has the remainder of 1 when it is divided by 23. This is very closely connected with the fact that the remainders must from a certain point on, repeat. Another way of expressing this result is that one can form a number completely of 9's, like 99999999, which is divisible by 23.

Exercise

Complete the following table. In doing this notice that it is not necessary to divide 10^{10} by 17 to get the remainder when it is divided by 17. We can compute each entry from the one above, like this: 10 is the remainder when 10 is divided by 17; this is the first entry. Then divide 10^2, that is, 100 by 17 and see that the remainder is 15. But we do not need to divide 1000 by 17. We merely notice that 1000 is 100 x 10 and hence the remainder when 1000 is divided by 17 is the same as the remainder when 15 x 10, or 150 is divided by 17. This remainder is 14. To find the remainder when 10^4 is divided by 17, notice that 10^4 is equal to 10^3 x 10 and hence the remainder when divided by 17 is the same as when 14 x 10 is divided by 17, that is 4. The table then gives the remainders when the powers of 10 are divided by various numbers:

	3	7	9	11	13	17	19	21	37	101	41
1	1	1	1			1					
10^1	1	3	1			10					
10^2	1	2	1			15					
10^3	1	6	1			14					
10^4	1	4	1			4					
10^5	1	5	1			6					
10^6	1	1	1			9					
10^7	1		1			5					
10^8	1		1			16					
10^9	1		1			7					
10^{10}	1		1			2					
10^{11}	1		1			3					
10^{12}	1		1			13					
10^{13}	1		1			11					
10^{14}	1		1			8					
10^{15}	1		1			12					
10^{16}	1		1			1					

Find what relationships you can between the number of digits in the repeating decimals for 1/3, 1/7, 1/9, 1/11, 1/13, etc. and the pattern of the remainders. Why does the table show that there will be five digits in the repeating portion of the decimal for 1/41? Will there be some other fraction 1/? which will have a repeating decimal with five digits in the repeating portion? How would you find a fraction 1/? which would have six digits in the repeating portion?

If you wish to explore these things further and find that you need help, you might begin to read some book on the theory of numbers. Also there is quite a little material on tests for divisibility in "Mathematical Excursions" by Miss Helen Abbott Merrill, Dover (1958).

continued from page 5.

Ex. 13. Show why the remainder when the sum of the digits of a number is divided by 9 is the same as the remainder when the number is divided by 9.

UNIT V

THE NON-NEGATIVE RATIONAL NUMBERS

1. Whole numbers and divisibility. You are familiar with the natural numbers 1, 2, 3, 4, 5, 6, and so on, and the number zero. These we have agreed to call the whole numbers. Later on we shall have what we call "negative numbers" as well and find that another name for the whole numbers is: "the non-negative integers", or "the positive integers and zero". But in this unit we shall just use the words "whole numbers" to describe the set 0, 1, 2, 3, etc.

Division is the "inverse" of multiplication; that is, 6/2 = 3 and 6 = 2 x 3 are two ways of expressing the same relationship. Also 6/4 = $1\frac{1}{2}$ because 6 = 4 x $1\frac{1}{2}$. When we divide 6 by 2, we obtain a natural number and we say "6 is divisible by 2". But when we divide 6 by 4 we do not obtain a natural number and we say "6 is not divisible by 4". The term "divisible" does not mean merely "you can divide" (this can usually be done - certainly in both cases above) but it means that both the divisor and quotient are natural numbers. Two other ways of saying "40 is divisible by 5" are "40 is a multiple of 5" and "5 is a factor of 40".

2. The fractional notation. The symbol 6/2 could have two meanings. It might be six halves or half of six, that is 6 x $\frac{1}{2}$ or $\frac{1}{2}$ x 6. The fact that these two are equal is called

the commutative property of multiplication. We see that six halves are 3, which is half of six.

$$\left(\tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2}\right) = 1 + 1 + 1 = 3.$$

So we use the two meanings interchangeably.

1/4 is the number such that if you multiply it by 4, you obtain 1; that is 4x(1/4)=1. In a similar way, 20x(1/20) or (1/20)x20=1.

3/4 would be 3x(1/4) or (1/4)x3. We can also say that 3/4 is the number such that if you multiply it by 4, you obtain 3.

The quotient of any two whole numbers we call a rational number whenever the quotient has meaning. Some examples are: 3/4, 7/2, 6/1, 125/789, 670000/3.

Exercises A

1. Give examples of the following kinds of numbers:
    - (a) natural numbers
    - (b) whole numbers
    - (c) non-negative numbers
    - (d) rational numbers
2. Express the relationships as products:
    - (a) (8/2)=4
    - (b) (21/7)=3
    - (c) (150/15)=10
    - (d) (29/5)=5 4/5
3. By what natural numbers is 144 divisible? What numbers are factors of 144?
4. List 10 multiples of 7.
5. List 5 multiples of 13.
6. What are two of the meanings of 3/8?

7. 2/3 is the number such that if it is multiplied by 3, 2 is obtained. Use this language to describe the following numbers:

   (a) 4/5 (b) 7/3 (c) 1/8 (d) 6/11 (e) 100/9

8. Using the fact that: 2x3x5x17=510, answer the following:

   (a) What numbers are factors of 510?

   (b) 510 is divisible by what numbers?

   (c) 510 is a multiple of what numbers?

9. See Exercise 8. By which of the following numbers is 510 divisible?

   4, 6, 8, 10, 11, 15, 20, 34, 51, 52

10. Assume a,b,c, and d are natural numbers. If axbxc=d, make as many statements as you can about factors and multiples involving 2 or more of the numbers a,b,c, and d. Is d a multiple of axb? Is bxc a factor of d?

3. Multiplication of rational numbers. In order to use rational numbers we must be able to multiply and add them, and we should like the properties of multiplication and addition to be the same as far as possible for the rational numbers as for the whole numbers. Since multiplication is a little easier than addition, we shall consider it first. What should be the value of (1/3)(1/4)? It is one-third of one-fourth. In other words we would divide something into four equal parts and then each of these parts into three equal parts. We would

have in all 12 equal parts. Hence we should define the product $(1/3) \times (1/4)$ to be $1/12$. Similarly, $(1/6) \times (1/5) = 1/30$. This would suggest what the product should be for any natural numbers in place of 3 and 4. One way to express this would be to replace 3 and 4 by letters instead of other numbers and have it understood that the letters stand for any numbers. Then we would have

$$(1/a)(1/b) = 1/(ab),$$

where ab means the product of a and b.

Suppose we have two rational numbers whose numerators are not 1, such as $(3/4) \times (5/7)$. Then this could be written

$(3/4) \times (5/7) = 3 \times (1/4) \times 5 \times (1/7)$ by the definition of a rational number and the associative property.

$= 3 \times 5 \times (1/4) \times (1/7)$, by the commutative principle.

$= 15 \times (1/28) = 15/28$, using the value of the product of two rational numbers with one in each numerator.

Would this work equally well with any natural numbers in place of 3, 4, 5, and 7? Expressed in letters would it be

$$(a/b)(c/d) = (ac)/(bd)?$$

In words what does this mean?

Exercises B

1. Explain what is meant by each of the following: 7/12, 5/3, 10/6, 14/24.
2. Calculate each of the following products: (5/3) x 6, (7/4) x 4, (3b/b) x b for several natural numbers in place of b.
3. We know that 6/6 = 1, 20/20 = 1. Using this and assuming that the product of any rational number and 1 is the rational number, find the value of:

   (3/5) x (6/6), (7/10) x (20/20), (11/8) x (7/7).
4. Compute the products indicated in the previous exercise, using the definition of the product of two rational numbers.
5. Can the natural number 6 be thought of as the rational number 6/1? Why?
6. Calculate the products, using the method of the preceding page for the product of (3/4)x(5/7), and giving the reasons for each step:

   (a) (1/2)x(3/5)   (b) (2/3)x(3/4)
   (c) (5/6)x(8/9)   (d) (1/4)x(2/3)x(7/8)
7. Using the definition of the product of two non-negative rational numbers, is the set of non-negative rationals closed with respect to multiplication?

8. Find the following products:

(a) 6x(3/11) (b) (2/9)x4 (c) (1/3)x(1/4)x(1/5)

(d) (2/3)x(7/8) (e) (7/5)x(5/7) (f) (1/4)x(8)x(3/6)x(9/4)

9. Suppose two equal rational numbers have equal denominators. What can you say about their numerators? Suppose the equal rational numbers had equal numerators, what could you say about their denominators?

10. State in words the method of finding the product of two rational numbers.

4. Equality of rational numbers. How do we know that 6/2 and 12/4 are two ways of representing the same number? Are there different ways of representing any rational number? We know that the answer to this is "yes" since, for example 1/2= 2/4. Here it is helpful to make a distinction that we made for natural numbers: there is a difference between a natural number and the symbol used to represent it. We call the symbol, the "numeral". Here when we want to make a distinction, we call the symbol the "fraction". If we were going to be very particular we would have written in the last section: a fraction which represents the product of two rational numbers is one whose numerator is the product of the numerators and whose denominator is the product of the denominators of the fractions which represent the given numbers. This, of course, is being altogether too particular. But it is useful

at times to have the word "fraction" for the symbol. For instance, we could say that the two fractions 1/2 and 2/4 represent the same rational number and so we call them equal. Also we should probably speak of the numerator and denominator of a fraction but not of a rational number. But this is awkward, too, and there is not likely to be any confusion when we speak of the numerator of a rational number, if we realize that it may have several numerators and that we are merely referring to the way it is written at the time.

We saw in exercises 3 and 4 above that (3/5)x(6/6) is on the one hand equal to (3/5)x1 which should be 3/5. On the other hand, if we multiply the numbers, we obtain 18/30. So 18/30 should be equal to 3/5. We could have used any natural number in place of 6 and we could have seen, for instance, that

$$3/5 = 21/35.$$

In fact, no matter what natural number k is, it would be true that $3/5=(3 \times k)/(5 \times k)$. We can write this more briefly as

$$(3/5)=(3k/5k).$$

We can multiply the numerator and denominator of any fraction by a natural number without changing the value of the rational number which it represents. Is this still true if k is any rational number? Also, working from the right to the left, we can divide both numerator and denominator of any fraction by the same natural number without changing the rational number which it represents.

How do we find out whether two fractions represent the same rational number? Suppose we had 6/15 and 4/10, in which the numerator of one is not a divisor of the numerator of the other. One method would be to reduce each fraction to lowest terms, that is, in each fraction divide the numerator and denominator by any common factor. Then the statements that $6/15=2/5$ and $4/10 = 2/5$ show that the two given fractions represent the same rational number. Another way of showing them equal would be to equate each fraction to one whose denominator is the product of the given ones. That is

$$6/15 = 60/150 \text{ and } 4/10 = 60/150.$$

In the first case we multiplied the numerator and denominator by 10 and in the second case by 15. If we do this using letters it is easier to see what the result looks like in general. Let the fractions be a/b and c/d. Then

$$(a/b) = (ad/bd), \text{ and } (c/d) = (cb/db).$$

Now $bd = db$, by the commutative property and $cb = bc$. Thus if the fractions (that is, the rational numbers which they represent) are equal, then $ad = bc$. Also if $ad = bc$, the fractions will be equal.

Exercises C

1. Prove in two ways that each of the following pairs of fractions represent the same rational numbers: 6/21 and 10/35, 9/12 and 21/28.
2. In the second method above we tested the equality of the two fractions by making the denominators equal. Could we

have the numerators equal instead? If so, what would the conclusions have been?

3. Use the conclusion that $a/b = c/d$, when $ad=bc$ to decide whether the following sentences are true or false:

(a) $(2/3)=(20/30)$ (b) $(1/10)=(100/1000)$ (c) $(5/6)=(51/61)$
(d) $(4/5)=(7/8)$ (e) $(17/51)=(3/9)$ (f) $(1/2)x(3/4)=9/24$

4. Reduce the following to lowest terms:

(a) 100/300 (b) 50/250 (c) 8/36
(d) 96/108 (e) 121/143 (f) 1924/2036

5. Show that $(4/7)x(7/4)=1$ and that $(9/17)x(17/9)=1$.

6. Show that $(a/b)(b/a)=1$ if a and b are natural numbers. The fraction b/a is called the reciprocal of a/b.

7. Write the reciprocals of the following numbers:

(a) 2/3 (b) 10/11 (c) 29/3 (d) 99/100 (e) 10

8. Which of the following sentences are true and which are false? Give the reasons for your answers:

$3/(2+6)=1/(2+2)$ $3/(2x6)=1/(2x2)$
$3/(6+12)=1/(2+4)$ $3/(6x12)=1/(2x4)$

Just showing that the numbers are equal or not equal is not enough. The proper or wrong use of the fundamental properties of rational numbers and natural numbers should be stated.

9. Is the following statement true for every rational number: Given a rational number, the reciprocal of its reciprocal is the given number.

5. Division by zero. So far both the numbers appearing in the fractions considered have been natural numbers. Why did we fail to mention fractions like 3/0? We know that 3/2 was defined so that (3/2) x 2 = 3. So 3/0 would have to be defined, if at all, so that (3/0) x 0 = 3. This would seem peculiar since we know that any natural number multiplied by zero is zero. But we still might not be disturbed by this. Suppose we carry it a little further. Then [(3/0) x 0] x 3 = 3 x 3 = 9. But (3/0) x (0 x 3) = (3/0) x 0 = 3. Hence the assumption (3/0) x 0 = 3 either leads to 9 = 3 or that [(3/0) x 0] x 3 is not equal to (3/0) x (0 x 3) which would deny the associative property. Our only choice then is to exclude zero denominators.

Exercises D

1. Should 0/3 be included among the rational numbers? Why? If it should be included, what number would it have to be equal to?
2. Use the argument in the paragraph above to show a contradiction we would reach if 4/0 were defined to be 4.
3. The number 1 is what multiple of each of the following?

(a) 1/2	(b) 1/3	(c) 1/10
(d) 1/100	(e) 1/1000	(f) 1/1000000

4. Does the question: "The number 1 is what multiple of zero?" have any meaning? Why?

5. Could 0/0 be admitted to the family of rational numbers withwithout running into trouble? Why?

6. Division of rational numbers. We have seen that it is easy to multiply two rational numbers. How can we divide them? Suppose we consider the quotient: $(3/5)\div(4/7)$. There are two ways to find this quotient. In the first place, we know that 2/3 is that number which when multiplied by 3 gives 2. Hence if we are to find the quotient $(3/5)/(4/7)$ we must search for a number which, when multiplied by 4/7 gives 3/5. In other words, we want to start with 4/7 and by multiplying by a properly chosen rational number, arrive at 3/5. If x stands for the number we are seeking, then

$$(4/7)\cdot x=3/5$$

If we multiply each of these two equal numbers by 7/4, the reciprocal of 4/7, and use the relation that $(7/4)\cdot(4/7)=1$, we obtain: $x=(7/4)\cdot(3/5)$. Hence the number we are seeking is $(7/4)(3/5)=21/20$.

We see $(3/5)\div(4/7)=21/20$. To check, we find $(4/7)(21/20)=84/140=3/5$. 21/20 is the number which multilied by 4/7 gives 3/5.

We can think of the quotient $(3/5)\div(4/7)$ as a quotient of two rational numbers, or as a single fraction with the rational number 3/5 as the numerator and the rational number 4/7 as the denominator, $\frac{(3/5)}{(4/7)}$

Then another way to get the same result is to notice that we can make the denominator of the given fraction 1 by multiplying numerator and denominator by the reciprocal of 4/7, that is, by 7/4. Then we have

$$(3/5)\Big/(4/7) = (3/5) \times (7/4) \Big/ (4/7)(7/4) = (3/5) \times (7/4) \Big/ 1$$

$$= (3/5) \times (7/4) = 21/20.$$

How would you formulate this in words? This shows that we have in the rational numbers a system which has one advantage over the natural numbers. The natural numbers are not closed under division, that is, the quotient of two natural numbers is not always a natural number. But the rational numbers are closed under division except by zero, since the quotient of any two rational numbers is a rational number, except when the divisor is zero.

Exercises E

1. What is the quotient of 3 divided by one-half? Find the result using division of fractions. Show how the same result could be obtained without dividing fractions.

2. Find the quotients of:

   (a) $(3/2) \div (9/4)$ (b) $(9/4) \div (7/6)$ (c) $(3/2) \div (7/6)$

   (d) $(5/6) \div 2$ (e) $3 \div (3/4)$ (f) $(10/11) \div (2/5)$

3. Find the quotients of:

   (a) $(3/2) \div \left[(9/4) \div (7/6)\right]$ (b) $\left[(3/2) \div (9/4)\right] \div (7/6)$

4. Is division of rational numbers associative?

5. Find the quotient:

$$(a/b) \div (c/d)$$

6. State the results in Exercise 5 in words.

7. Addition of rational numbers. We have seen how to multiply rational numbers. How do we add them? If the denominators are the same, it is easy. For instance

$$3/7 + 2/7 = 3 \times (1/7) + 2 \times (1/7) = (3 + 2) \times (1/7)$$
$$= 5 \times (1/7) = 5/7$$

We just assume that the distributive property will hold and define addition accordingly. We could express this in terms of letters:

$$a/c + b/c = (a+b)/c.$$

When the denominators are equal we add the numerators, retaining the common denominator.

Suppose we have two rational numbers whose denominators are not equal. Then we can make the denominators equal by multiplying numerator and denominator by appropriate natural numbers and then add the numerators. Suppose we wish to add 2/7 and 3/5. Then we have

$$2/7 + 3/5 = 10/35 + 21/35 = (10 + 21)/35 = 31/35.$$

We chose the 35 as the denominator since it had to be a multiple of 7 and 5 and the smallest such number is 35. Why is this the smallest number? Suppose on the other hand, we were to add 3/4 and 7/10. Here our denominator must be a multiple of 4 and also of 10. While 40 satisfies these conditions, 20 is a smaller number which does. Thus the numbers could be written

$$3/4 + 7/10 = 15/20 + 14/20 = 29/20.$$

You may prefer to write this in column form as

$$\begin{array}{rl} 3/4=15/20 & \text{or } \ 15(1/20) \\ +\ 7/10=\underline{14/20} & \text{or } \ \underline{+14(1/20)} \\ 29/20 & \quad 29(1/20)=29/20 \end{array}$$

Exercises F

1. Find the sums:

   (a) 7/8 + 3/8 (b) 3/5 + 6/5 (c) 7/8 + 3/16

   (d) 7/8 + 3/5 (e) 7/8 + 9/21 (f) 7/8 + 11/20

   (g) 11/12 + 3/24

2. Find the value of the following:

   (a) 1 ÷ (1/3 + 1/5) (b) (3+5) ÷ (1/3 + 1/5)

3. We defined addition so that the numerator of the sum of two rational numbers having equal denominators was obtained by adding the numerators, the common denominator being retained. Is the sum of two rational numbers having equal numerators the fraction whose numerator is the common numerator and whose denominator is the sum of the denominators; that is, is 5/7 + 5/3 = 5/10? Give reasons.

4. Find the value of (8/13) - (2/7).

5. How would you subtract one rational number from another?

6. If possible using non-negative rationals, subtract from 7/8 the following:

   (a) 1/4 (b) 2 (c) 3/4 (d) 8/9 (e) 13/16

7. Find the value of (7/4) $\left[(6/7) + (9/8)\right]$

8. Find the values of:

   (a) 0/3 + 7/8 (b) 0/a + b/c

   (c) (0/3) x (7/8) (d) (0/k) x (a/b)

9. Find: (a/b) + (c/d)

10. Why should finding the least common multiple of the denominators of two fractions be useful in finding the sum of two rational numbers?

11. Is every whole number a rational number? Why?

12. Two fractions 0/a and 0/b are equal when a and b are any natural numbers since 0 x b = a x 0. Also 0/a is zero since (0/a) x a = 0 and 0 x a = 0. Show that if the must be zero. (Do you already know that this property holds for the whole numbers?)

property holds for the whole numbers.)

8. Summary of the properties of the non-negative rational numbers. It is probably worth while to list the properties which we have found so far. The rational numbers are represented by "ordered pairs" of numbers the first of which is a whole number and the second of which is a natural number. We call them "ordered pairs" since the order in which they are written is important; that is 3/4 is not the same number as 4/3. We use the solidus (the name for the slanting line) to separate them. But we could write 3,4 or $\frac{3}{4}$ or 3*4 just as well. We defined equality, sum and product and they have the following properties:

1. (closure) The product and sum of any two rational numbers are rational numbers.

2. (existence of identity number for addition and multiplication). The number 0 is a rational number and has the property that $0 + r = r$ for any rational number, r; zero is the identity number for addition. The number 1 is a rational number and has the property that $1 \times r = r$ for any rational number, r; one is the identity number for multiplication.

3. Addition and multiplication are associative.

4. Addition and multiplication are commutative.

5. The distributive property holds.

6. The quotient of any two rational numbers is a rational number if the divisor is not zero.

7. If the product of two rational numbers is zero, one or both must be zero.

8. Zero multiplied by any rational number is zero.

Exercise G

Which of these properties are also properties of the set of whole numbers?

9. Ordering of rational numbers. Let us first review a few facts about the whole numbers. We are familiar with the notation $7 - 5 = 2$. This is just another way of writing $7 = 5 + 2$. In words, $7 - 5$ is the number which, when added to 5 gives 7. Now $5 - 7$ is not a whole number since there is no whole number which we can add to 7 to get 5. Similarly,

18 - 10 is a whole number but 10 - 18 is not. In general, one natural number minus another natural number is a natural number only if the first is greater than the second. There is a notation for this: $7 > 5$ or $18 > 10$ means "7 is greater than 5" or "18 is greater than 10". We could also say "5 is smaller than 7," written $5 < 7$; or "10 is smaller than 18," written $10 < 18$. This could be written in terms of letters as follows:

$b - a$ is a natural number if $b > a$, that is $a < b$.

These same symbols of inequality are useful in dealing with rational numbers. Suppose we wish to compare 1/3 and 2/7; which is greater? One way of doing this would be to find their decimal equivalents; this we shall do in the next section. The second way, which is probably simpler, is to replace the pair of fractions by a pair with the same denominator just as if we were going to add them. That is, $1/3 = 7/21$ and $2/7 = 6/21$. Since 7 is greater than 6, this shows that 1/3 is greater than 2/7. Another way to look at it is to see that $1/3 - 2/7 = 1/21$ which is the quotient of two natural numbers. In general, one rational number is said to be greater than a second rational number if the first minus the second is the quotient of two natural numbers; another way to say it would be: one rational number is greater than a second if one can add the quotient of two natural numbers to the second to get the first.

Notice that we have used "the quotient of two natural numbers". Why did we not just say "rational number"? One reason is that zero is a rational number and if their difference were zero, they would be equal. Also we shall later be considering negative rational numbers, and we wish to exclude them from our definition.

Exercises H

1. Associate 7/10, using the appropriate symbol <,=,> with each of the following:
   9/10, 1/10, 1/2, 3/5, 3/4, 3/7, 0, 21/30, 7/5, 7/6, 7/8, 7/9, 7/11, 7/12
2. If two rational numbers have the same denominator, the larger rational number has the larger numerator. If two rational numbers have the same numerator show that the larger rational number has the smaller denominator.
3. Write the following rational numbers in increasing order: 8/9, 18/19, 3/4, 5/6, 25/27
4. Write all the fractions between 0 and 1 whose denominators are 7 or less, in increasing order of size. There are a number of interesting properties of this set of numbers. Can you discover them?
5. If a, b, c, d are natural numbers show that $(a/b)>(c/d)$ if $ad > bc$. Show also that if $ad < bc$ then $(a/b)<(c/d)$. How could this be used to shorten the computation above?

6. If the diameter of a circle is 1, it is shown in geometry that the number of units in the circumference is a number designated by π and whose value to five decimal places is 3.14159. The rational number most often used as an approximation for this number is 22/7. Another approximation used by the Babylonians is 355/113. Which of these fractions is the greater and which is closer to π ?

7. If a and b are two whole numbers, then just one of the following relationships holds: $a > b$, $a = b$, $a < b$. Show that the same statement may be made when a and b are rational numbers.

*8. (Hard) Let r and s be two positive rational numbers with $r < s$. Show each of the following for two pairs of values of r and s. For example, use $r = 1/3$ and $s = 2/5$.

   a. $r < \left[(r + s)/2\right] < s$.
   b. $1/s < \left[(1/r + 1/s)/2\right] < 1/r$.
   c. $r < \dfrac{2}{1/r + 1/s} < s$.
   d. if $r = a/b$ and $s = c/d$, then $r < (a+c)/(b+d) < s$.

9. What part or parts of exercise 8 show, if they are true in general, that between any two rational numbers there is another rational number?

* The notation in the parts of the exercise perhaps need further explanation. We write $2 < 4 < 7$ to mean "2 is less than 4 and 4 is less than 7" or, more briefly, "4 is between 2 and 7 and equal to neither". The corresponding meaning would be used for rational numbers.

10. (Very hard) Show that the inequalities of exercise 8 hold for all positive rational numbers r and s. Begin by letting $r = a/b$, $s = c/d$, where a, b, c, d are natural numbers.

10. Decimal equivalents of rational numbers. We saw above that one way to compare the size of 1/3 and 2/7 was to compare their decimal equivalents. To limber up our pencils and our minds, let us start by finding a few decimal equivalents.

Exercises I

1. Find the decimal equivalents to ten places of each of the following: 1/2, 1/3, 1/4, 1/5, 1/6, 1/7, 1/8, 1/9, 1/10, 1/11.

2. Point out any patterns which you see in the decimal equivalents which you have just calculated. In particular was there any stage at which you could write down the answer without carrying the actual division farther? Which of the decimal equivalents were exact?

First of all let us look at these decimal expansions which are exact, that is, which end with a string of zeroes. We had 1/2 = .5, 1/4 = .25, 1/5 = .2, 1/8 = .125, 1/10 = .1. This kind of decimal is sometimes called a terminating decimal since it stops. Instead of using the decimal notation we could have used fractions. Then we would have 1/2 = 5/10, 1/4 = 25/100, 1/5 = 2/10, 1/8 = 125/1000, 1/10 = 1/10.

3. Express each of the following decimals as a quotient of two whole numbers where the denominator is a power of ten, that is, one of 10; 100; 1000; 10000; etc: 15.78, 1.7893, .0012.

4. Do you believe that every terminating decimal can be expresses as the quotient of two whole numbers in which the denominator is a power of ten? Why?

5. Express each of the following as a decimal: 156/1000, 57/10000, 789/100, 3589/10. Do you believe that any quotient of two whole numbers in which the denominator is a power of ten, can be expressed as a terminating decimal? Why?

6. What connection is there between the answers for exercises 4 and 5 above? Collect these results into a single statement if you can.

7. The fraction 1/8 could be written as a terminating decimal, as we saw above, because it can be written as a fraction whose denominator is 1000, namely, 125/1000. 1/25 could be written as a terminating decimal because it is equal to 4/100. Is there any way that one can determine when a rational number has a terminating decimal without converting it to a fraction with a power of ten as denominator?

We saw from exercise 7 and other examples that if the fraction a/b is to have a terminating decimal it must be equal to a fraction c/d where d is a power of 10. Now if a/b is in lowest terms, it can be equal to c/d only if b divides d. In other words b must divide a power of 10. For example, 8 must divide 1000 so that 1/8 can be equal to 125/1000, 25 must divide 100 so that 1/25 is equal to 4/100.

8. If a natural number is a divisor of a power of 10 what can you say about its prime factors? (The teacher should recall to the student what is meant by "prime factors").

9. If a number has no prime factors but 2 or 5 or both, must it be a divisor of a power of ten? Illustrate your conclusion with several examples.

So we can summarize what we have found so far by the following statement: If a rational number has a terminating decimal equivalent and if the fraction is in lowest terms, then the only prime factors of the denominator can be 2 or 5 or both. Conversely, if the denominator of a fraction in lowest terms has no prime factors but 2 or 5 or both, then its decimal equivalent terminates.

11. Repeating Decimals. In the first exercise of the preceding section we found that there were several fractions whose decimal equivalents did not terminate: 1/3, 1/6, 1/7, 1/9, 1/11. These do not terminate since the denominators have factors different from 2 and 5. Next we look into these in more detail.

One way to write a decimal equivalent for 1/3 would be $.3333\underline{3}\ldots$ where the line under the 3 and the three dots afterward indicate that no matter how far out one carries the division there will be just a sequence of threes. Similarly 1/11 could be written $.0909\underline{09}\ldots$ where it is the pair of digits 09 which repeat as far as the division is carried out. Also 4/3 can be written $1.33\underline{3}\ldots$. The fraction 1/7 has a repeating portion of six digits: $.\underline{142857}\ldots$. Such decimals as these are called *repeating decimals* (sometimes, periodic decimals). That is, a decimal is called a repeating decimal when from a certain point on some sequence of digits repeats and continues to repeat no matter how far the division is carried out. Notice that $1.33\underline{3}\ldots$ is a repeating decimal even though the initial digit is not 3. Similarly, $14.235\underline{235}\ldots$ is a repeating decimal.

These decimals which we have found for 1/3, 1/7, etc., do not give the exact value for the fraction no matter where one cuts them off but the farther one goes, the closer is the decimal in value to the number. For instance 1/3 - .3 = 1/3 - 3/10 = 1/30, 1/3 - .33 = 1/3 - 33/100 = 1/300, 1/3 - .333 = 1/3 - 333/1000 = 1/3000 and so on. The results of Exercise 2 below show similar results for the expansion of 1/7. For instance, .142857 is equal to 142,857/1,000,000 but this is not equal to 1/7 since 7 times 142,857 is 999,999 which is just short of 1,000,000. However 1/7 - .142857 = 1/7000,000 which is a very small number.

Exercises J

1. Sometimes one writes 1/3 = .33 1/3. What does the second 1/3 stand for? Is it the same as the first 1/3? Is the following true: 1/3 = .333 1/3? If so, what does the second 1/3 stand for here?
2. What would one have to add to .142 to make it exactly equal to 1/7? What would one have to add to .1428 to make it exactly equal to 1/7?
3. Multiply each of the following by 3: .3, .33, .333, .3333. By how much does each of your results differ from 1? What connection is there between your answers here and exercise 1 above?
4. Find the decimal equivalents for each of the following fractions. (Do not be discouraged at the size of the denominator in some cases. The process for some large denominators is shorter than for smaller ones.) Carry out the division to the point where the decimal terminates or begins to repeat.

   (a) 3/8 (c) 15/37 (e) 41/333 (g) 1/13
   (b) 5/44 (d) 7/125 (f) 4115/33,333 (h) 1/17 .
5. In the decimal equivalents for 1/3, 1/7, 1/13, 1/17 do you see any connection between the denominator and the number of digits in the repeating part of the decimal?

6. Do not carry out the division for 5/413 but guess whether or not the decimal will terminate or repeat. Give reasons for your guess. How far might you have to carry out the division to show your guess to be correct or false?

7. How many different remainders would it be possible to have in dividing a number by 727? What would a remainder have to be if the decimal terminates?

12. Rational Numbers Equivalent to Repeating Decimals. It is a remarkable fact that the decimal equivalent of every rational number either terminates or is a repeating decimal. You may have guessed this already. To see why it is so, consider first a few divisions. First take 4/15 (this division is to be written out). Here the remainder after two divisions is the same as after three and the process just repeats itself. Consider the decimal for 2/7 (this division is to be written out). Here the first remainder is 6 and the remainder after six more divisions is 6, which means that the series repeats. Consider 575/17 (this division to be written out). The first remainder is 6, the second is 14, the third is 4 and the fourth 6. It does not at this point begin to repeat since the first 6 occurred before zeros were adjoined. But as soon as a third 6 occurs as a remainder the decimal will begin to repeat. As a matter of fact, the remainder 14 is the first one to occur again and the decimal will start to repeat at this point.

Thus in finding the decimal equivalent of any rational number either the decimal will terminate or, from a certain point on, one must continue to adjoin zeros to the dividend. If, after zeros are adjoined, two remainders are the same the decimal begins to repeat and continues to do so. Why must two of these remainders be equal? The last two exercises in Exercises J should be a help in reaching the answer to this question.

In division by 17, the only possible remainders would be 0, 1, 2, 3, ... , 15, 16. If a remainder is zero, the decimal will terminate. From what we have shown above, this would not happen if the fraction were in lowest terms since the denominator has factors other than 2 and 5. If the decimal does not terminate, there would be only 16 possible remainders. Suppose in finding the decimal equivalent of 1/17 the first sixteen remainders were all different (we saw above that this was indeed the case). Then the next one would have to be a remainder that had occurred before. Similarly, in dividing by 37 there would be not more than 36 possible remainders and if the first 36 were all different, the next one would have to be one which had already occurred. Actually the first three remainders in computing 42/37 are 5, 13, 19, and the fourth remainder is 5 again. Since the first remainder occurred just before a zero was adjoined to the 42, the decimal repeats from the fourth remainder on. It is 1.135135 ... . Hence it is not necessary that all the remainders occur before repetition. But we can be sure in every case that the largest possible number of digits in the repeating part of any repeating decimal is one less than the divisor.

This discussion shows that every rational number has a decimal equivalent which either terminates or is a repeating decimal.

So far we have considered converting rational numbers into their decimal equivalents. Suppose we have a repeating decimal: $.12\underline{12}$ ... . Can we find a rational number which it represents? Before trying this let us go back to one which we already know and develop a method for dealing with it so that we may apply it to the case at hand.

Consider the decimal: $.33\underline{3}$ ... . Let the letter n stand for this number. Then ten times this number, that is 10n will be $3.33\underline{3}$ ... . That is, we have

$$10n = 3.33\underline{3} \ldots$$

$$n = .33\underline{3} \ldots$$

If we subtract n things from 10n things we have 9n things. (This can also be seen from the distributive property: $10n - n = (10 - 1)\, n = 9n$). And $.33\underline{3}$ ... subtracted from $3.33\underline{3}$ ... is 3.000. Hence we have $9n = 3$. But, using our notation for a rational number, we see that this means $n = 3/9$ which is equal to 1/3. This is a complex way of showing that 1/3 has the decimal equivalent given above but it is useful to look at this process since it will apply for more difficult decimals.

Now let us return to $.1212\underline{12}$ ... . Try instead 100n = $12.12\underline{12}$ ... . Then 99n = 12.000 and n = 12/99 which reduces to 4/33.

We do not attempt to give a formal proof that every repeating decimal represents the quotient of two natural numbers, that is, a rational number, but working the exercises which follow should be evidence in that direction.

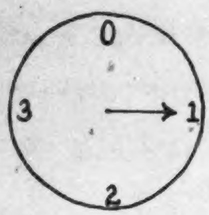
Exercises K

1. Find the rational number whose decimal equivalent is $.1212\underline{12}$ ... . Find the rational number whose decimal equivalent is .121121121 ... .
2. Express each of the following in the form a/b where a and b are integers: $.3434\underline{34}$ ... , $1.3434\underline{34}$ ... , $13.4343\underline{43}$ ... , $.567\underline{567}$ ... , $1.23412341\underline{234}$ ... , $5761.23123\underline{123}$ ... .
3. Can you formulate any rule for determining what n is to be multiplied by in dealing with such repeating decimals?
4. Look again at the number of digits in the repeating parts of the decimal equivalents for 1/3, 1/7, 1/11, 1/13, 1/17. We have seen above that the number of digits in the repeating part cannot be as great as the denominator. Can you discover any sharper relationship?

UNIT XIV

MATHEMATICAL SYSTEMS

A New Arithmetic



1 + 2 = 3

2 + 3 = 1

2 + 2 = 0

Here are some new number facts. Do they seem a little strange to you? We might call this "clock arithmetic." We have used a four-minute clock -- one which might be used to time rounds and intermissions in a boxing match.

Let's see how this kind of arithmetic works. If the hand is at 1 minute, and moves for 2 minutes, then it is at 3. We can write 1 + 2 = 3. If it is at 2 and moves for 2 minutes then it is at 0. We write 2 + 2 = 0. If it is at 2 and moves for 3 minutes, then it stops at 1. We write 2 + 3 = 1.

We can make an addition table for this system of arithmetic thus:

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

We read tables of this sort by following across horizontally from any entry in the left column, say 2, to the position below some entry in the top row, say 3. The entry in this position in the table(s) is then taken as the result of combining the element in the top row with the element originally picked out in the left hand column. In the case above we write 2 + 3 = 1. Check that 3 + 1 = 0 in this table. Studying our table, is 1 + 2 = 2 + 1? is 2 + 3 = 3 + 2? What does this suggest to us about this kind of arithmetic?

Is $(1 + 2) + 3 = 1 + (2 + 3)$?

Is $1 + (1 + 2) = (1 + 1) + 2$?

Check some other examples. What does this suggest to us?

Let's compare this new arithmetic with ordinary arithmetic.

1. What are the numbers of the ordinary arithmetic? Of this new arithmetic?
2. Does addition have the commutative property in this new arithmetic?
3. Does addition have the associative property in this new arithmetic?
4. Is there an identity element (an element which when combined with any other element produces the "other" element itself as the result) for addition in this new arithmetic?

We call this new kind of arithmetic "modular arithmetic", and the number 4 is called the modulus. We say this system is arithmetic mod 4. The arithmetic of the three-minute egg timer is arithmetic mod 3. We can write an addition table for mod 3, mod 5, mod 8 -- we can have as many modular arithmetics as we have natural numbers.

Exercises - 1

1. Make an addition table for mod 3, mod 5, mod 6, and mod 8. What are the numbers for each? Save for use in Exercises 8.
2. Using the mod 5 addition table, find simpler names for:

$1 + 2$

$3 - 1$

$0 - 3$

$4 + 1$

$3 + 2$

$3 + 4$

$1 - 2$

$3 - 2$

$2 - 4$

$4 + 0$

$(3 + 2) + 4$

$3 + (2 + 4)$

$(4 - 2) + 3$

$(3 - 1) + (2 + 4)$

$4 - (2 - 3)$

$(3 + 4) - 3$

$(2 - 3) - 3$

$(1 - 4) + 4$

$3 + (3 - 1)$

$(3 + 2) + (2 - 1) - (4 - 3)$

3. You have only a five-minute clock. How many full turns would the hand make if you were using it to tell you when 23 minutes had passed? Where would the hand be at the end of the 23 minute interval? If you continue then for 15 minutes where is the hand?

Can you figure out an easy way to work problems like this without counting on the clock? Try it.

What is an Operation?

We are familiar with the operations of ordinary arithmetic -- addition, multiplication, subtraction, and division. In the preceding exercises we did a different kind of operation. We made a table for the new addition. This operation is defined by the table, because it tells us what we get when we put two numbers together. Study the following tables.

(a)

+	1	2	3	4	5
1	2	3	4	5	1
2	3	4	5	1	2
3	4	5	1	2	3
4	5	1	2	3	4
5	1	2	3	4	5

(b)

+	3	5	7	9
3	6	8	10	12
5	8	10	12	14
7	10	12	14	16
9	12	14	16	18

(c)

+	1	2	3	4	5	6	7	8	9	10	11	12
1	2	3	4	5	6	7	8	9	10	11	12	1
2	3	4	5	6	7	8	9	10	11	12	1	2
3	4	5	6	7	8	9	10	11	12	1	2	3
4	5	6	7	8	9	10	11	12	1	2	3	4
5	6	7	8	9	10	11	12	1	2	3	4	5
6	7	8	9	10	11	12	1	2	3	4	5	6
7	8	9	10	11	12	1	2	3	4	5	6	7
8	9	10	11	12	1	2	3	4	5	6	7	8
9	10	11	12	1	2	3	4	5	6	7	8	9
10	11	12	1	2	3	4	5	6	7	8	9	10
11	12	1	2	3	4	5	6	7	8	9	10	11
12	1	2	3	4	5	6	7	8	9	10	11	12

(d)

▱	0	1	2	3
0	0	1	2	3
1	1	3	4	5
2	2	5	6	7
3	3	7	8	9

(e)

θ	1	2	3
1	3	1	2
2	1	2	3
3	2	3	1

We see that these tables show us a way to put two things together to get one and only one thing. For example,

2 + 2 = 4 in both tables (a) and (c)

2 ▱ 1 = 5 and 2 ▱ 2 = 6 in table (d)

1 θ 1 = 3

When we have a way of putting two things of a given set together to get a third, we say we have a binary operation. For instance,

8 and 2 when added gives us 10.

8 and 2 when multiplied gives us 16.

8 and 2 when "sumptified" gives us 18. When 6 and 4 are sumptified we get 16. 5 and 1 when sumptified give us 11. Were any of the tables a "sumptification" table?

This doesn't mean that we can always put things together in any order. For example,

2 ▱ 1 = 5, but 1 ▱ 2 = 4

For this reason, we must remember first that when we explained how to read a table we decided to write the element in the left hand column first and the element in the top row second with the operation's symbol between them. We must then remember to examine each new operation to see if it is commutative and associative.

Exercises - 2

1. Use the tables of operation on page 3 to answer these questions.

A	B
(a) 3 + 3 = ? if we use table (a)	(a) 12 + 12 =
(b) 3 + 3 = ? if we use table (b)	(b) 10 + 6 =
(c) 3 ▱ 2 =	(c) 3 θ 1 =

(d) 2 ⊖ 2 =

(e) 1 ⊖ 1 =

(f) 11 + 12 =

(d) 1 ▱ 3 =

(e) 1 ▱ 1 ▱ 2 =

(f) 2 ⊖ 3 ⊖ 3 =

2. Which of the binary operations described in the tables on page 3 are commutative? associative? Is there an easy way to tell if an operation is commutative when you examine a table of operations? What is it?

3. Are the following binary operations commutative? associative?

   (a) Set: All natural numbers less than 50.

   Operation: Twice the first added to the second.

   Example: 3 combined with 5 produces 11 (2 x 3 + 5 = 11)

   (b) Set: All natural numbers between 25 and 75.

   Operation: Choose the lesser number.

   Example: If the two numbers are 28 and 36, the third number associated with them by this operation is 28.

   (c) Set: All natural numbers between 500 and 536.

   Operation: Choose the greater number.

   Example: If the two numbers are 520 and 509, the third number is 520.

   (d) Set: The prime numbers

   Operation: The larger number.

   (e) Set: All natural numbers.

   Operation: Least Common Multiple.

   Example: If the numbers are 4 and 6, the third number determined by this binary operation is 12.

   (f) Set: All natural numbers.

   Operation: Greatest Common Factor.

   (g) Set: All natural numbers.

Operation: Given two natural numbers, m and n, the result of the operation is m^n.

(h) Set: The prime numbers.

Operation: The larger number.

(i) Set: Gallon cans of paint in different colors.

Operation: Mixing paint.

4. Make up a table for an operation that has the commutative property.
5. Make up a table for an operation which does not have the commutative property.

More about Closure

We already have an acquaintance with the idea of closure. What do you remember from that brief introduction?

We recall that a set is closed under an operation if we can always do that operation on any two members of the set and get a unique third number which is a member of the same set. The two members we start with may be the same one. For example,

(1) We observed that the set of even numbers is closed under addition. This means that if we add any two even numbers, we get a third even number.

$2 + 2 = 4$ (We used the same number.)

$14 + 6 = 20$

$44 + 86 = 130$

(2) We observed that the set of odd numbers is not closed under addition. This means that if we add two odd numbers we do not get a third odd number. For example, $3 + 5 = 8$. Is this a case where one example is enough to show that closure does not hold? We can actually give more examples. The sum of two odd numbers is always outside the set of odd numbers.

(3) We observed that the set of natural numbers is not closed under subtraction, that is, take a pair of numbers, 6 and 9, and subtract. 6 - 9 -- but no natural number has the name, "6 - 9." Yet, we can subtract 6 from 9 to give us the number, 9 - 6. The standard symbol for this number is "3".

(4) We observed that the set of natural numbers is not closed under division. It is true that $\frac{8}{2}$ is a natural number, but there is no natural number, $\frac{9}{2}$. What are some other illustrations of closure, that is, sets closed under an operation and sets not closed under an operation?

Exercises - 3

1. Study again the tables on page 3. Which sets are closed under the operation? Which sets are not closed under the operation? How do you know?

2. Which of the systems described below are closed?

   (a) The set of even numbers under addition.

   (b) The set of even numbers under multiplication.

   (c) The set of odd numbers under multiplication.

   (d) The set of odd numbers under addition.

   (e) The set of multiples of 5 under addition.

   (f) The set of multiples of 5 under subtraction.

   (g) The set of even numbers used in telling time under clock addition.

   (h) The set of odd numbers used in telling time under clock addition.

   (i) The set of numbers mod 7 under subtraction.

   (j) The set of natural numbers less than 50 under the operation where the third number is the smaller of the two numbers.

(k) The set of prime numbers under addition.

(l) The set of numbers whose numerals in base 4 end in '0' or '2' under addition.

(m) The set of numbers whose numerals in base 5 end in '3' under addition.

Identities (or Identity Elements)

In our study of the number one in ordinary arithmetic, we observed that any number multiplied by 1 gave that same number, that is, the product of any number and 1 is the number, like,

$2 \times 1 = 2 \quad 3 \times 1 = 3 \quad 156 \times 1 = 156 \quad \frac{2}{3} \times 1 = \frac{2}{3}$ or, for any number in ordinary arithmetic, $n \cdot 1 = n$.

In our study of the number zero, we observed that the sum of 0 and any number in ordinary arithmetic gave the number, that is,

$2 + 0 = 2 \quad 3 + 0 = 3 \quad 468 + 0 = 468 \quad \frac{8}{5} + 0 = \frac{8}{5}$ or for any number in ordinary arithmetic, $n + 0 = n$.

One is the identity for multiplication in ordinary arithmetic.

Zero is the identity for addition in ordinary arithmetic.

What is the identity for the arithmetic of the 4-minute clock? for our ordinary clock?

What tables of operation in Exercises 1 have identities? What is the identity for each?

Inverses

If we add two things and get the identity for addition, then we call them additive inverses of each other. For example, in the table

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

0 is the identity.

$2 + 2 = 0$
$3 + 1 = 0$
$1 + 3 = 0$

These pairs of numbers, 2 and 2, 3 and 1, 1 and 3, are said to be inverses of each other. Each element of the set has an inverse. The inverse of 0 is 0, the inverse of 1 is 3, the inverse of 2 is 2, and the inverse of 3 is 1.

Exercises - 4

1. Study tables on page 3.
   - (a) Which tables have an identity and what is the identity?
   - (b) Pick out inverses in these tables. Does each member of the set have an inverse?

Some Algebraic Systems

We have an algebraic system when the following statements are true:

1. There is a set of things -- these things need not be numbers.
2. There is one or more operations.
3. There are some properties concerning the operations and the sets of things -- such as the commutative property, the associative property, closure, identities, inverses.

Let's look at egg-timer arithmetic -- arithmetic mod 3.

+	1	2	0
1	2	0	1
2	0	1	2
0	1	2	0

(a) It has a set of things. These are numbers -- 0, 1, 2.

(b) It has the operation, +.

(c) The operation of + has the commutative property. Can you tell by the table? If so, how? We can make some checks too. 1 + 2 = 0 and 2 + 1 = 0, so 1 + 2 = 2 + 1.

(d) There is an identity for the operation, + (the number 0).

(e) Every member of the set has an inverse for the operation +.

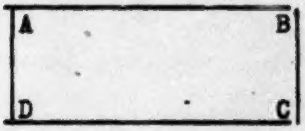
Algebraic Systems without Numbers

We may have algebraic systems without numbers in them. Suppose we invent one. What do we need?

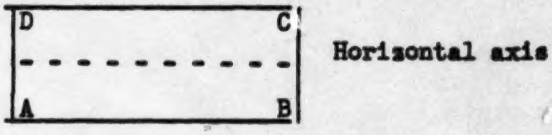
We must have a set of things. Then, we need some kind of operation -- something we can do with two of the things to get a third. And there must be properties concerning the operation and the things in the set.

Let's start with a post card -- really any rectangular shaped card will do. Instead of a set of numbers we will have a set of changes of position. We will take only those changes which make the card look like it did in the beginning (except that the marks on the corners may be moved around). How many of these changes are there?

We may start with it in some position which we will call the original position. We will say it looks like this:

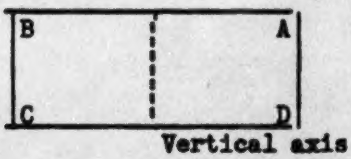


Letters in the corners of the card will help us see the different changes. One new position may be like this. The change of position was turning the card from its original position on its horizontal axis.



A second change of position is this:

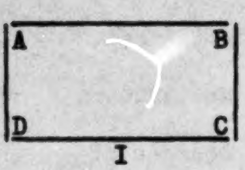
(Turn the card from its original position on its vertical axis.)



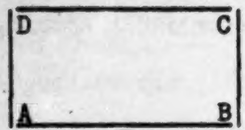
There is a third change -- we may turn the card from its original position halfway around its center. It looks like this:

Another change is to leave the card as it is.

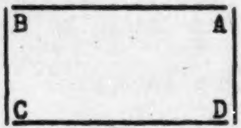
Here are the four changes of position:



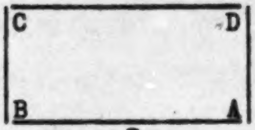
I



H



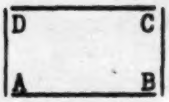
V



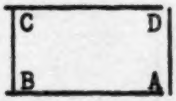
R

We can now make up our mathematical system. The set of things in our system is the set of changes I, H, V, and R. We will need an operation. Let's make up one -- it is *.

H * V means first do change H



Then, do change V



This final position is the same change as change R. Therefore, H * V = R

What shall we call this operation?

Complete this table:

*	I	H	V	R
I	I	H	V	_
H	H	_	R	_
V	_	_	I	H
R	_	_	H	I

Is this really an operation? What properties exist between the operation and set of things?

Exercises - 5

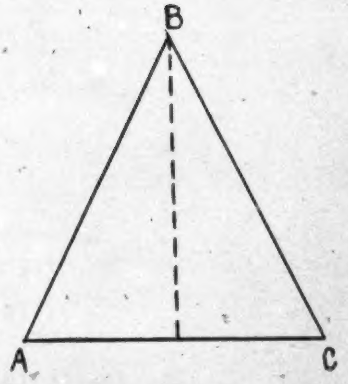
1. Examine the table of operations for the changes of the rectangle.

   (a) Is the set closed to this operation?

   (b) Is the operation commutative?

   (c) Is the operation associative?

   (d) Is there an identity for the operation?



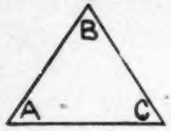
2. Here is another system of changes. Take a triangle with two equal sides. Label the corners "A", "B", "C", so it will look like this:

The set for the system will consist of two changes. The first change, called I, will be "leave alone". The second change, called M, will be "flip the triangle around its vertical axis."

$M^{\circ}I$ will mean flip the triangle about the vertical axis and then leave the triangle alone. How will the triangle look -- as if it had been left alone, I, or as if the change M had been made?

The operation, called $I^{\circ}M$ is: First do ______, and then do ______.

Always start the operation with the triangle in this position.



Does $M^{\circ}I = M$ or does $M^{\circ}I = I$?

(a) Complete the table below:

o	I	M
I		
M		

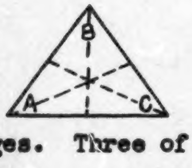
(b) Is the set closed for this operation?

(c) Is the operation commutative? associative?

(d) Is there an identity for the operation?

(e) Does each member of the set have an inverse for the operation?

3. Make a triangle with three equal sides. Label its corners "A", "B", "C", like this:



The set for this system will be made of six changes. Three of these will be flips about the axes, and three will be turning the triangle around its center.

Make a table for these changes. Examine the table. Is this operation commutative? Is there an identity change? Does each change have an inverse?

4. Try making a table of changes for a square.

There are eight changes. What are they? Is there an identity change? Is the operation commutative?

5. This is the table of changes of a triangle with two equal sides.

O	I	M
I	I	M
M	M	I

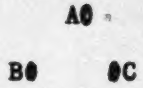
Fill in this table of operation for addition modulus two.

+	0	1
0		
1		

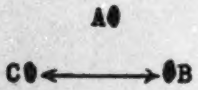
Suppose a "O" is put in place on every "I" in the table of changes, and a "1" is put in place of every "M" and a "+" is put in place of the "O". What would the resulting table be?

The two tables use different symbols, but have the same pattern. We may then expect them both to have the same properties.

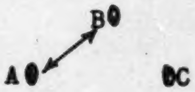
6. Another mathematical system which does not use numbers is the system of changing tires on a tricycle. Suppose tires on a tricycle are labeled like this:



By switching two tires we could get



or



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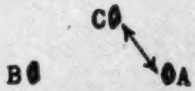
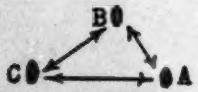


By switching all three tires we could get



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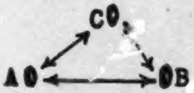

The set is made of tire switches, not tires.

(a) What will the operation be?

(b) Give a name to each of the five switches above. Let "I" be the "identity switch", that is the switch which makes no change at all. Make a table for the system of switching tires on a tricycle.

(c) Does this table have the same pattern as the table of changes for a triangle with three equal sides?

(d) Was the operation for changes of the triangle commutative?

(e) Is the operation of tire switching commutative? Check and see.

7. There are three pictures on the wall. We can leave them alone, switch two, or switch all three in various ways. See if you can make a system and a table for switching pictures. Remember to have a system you need a set of things, and an operation. What properties do you find in this system?

Algebraic systems may be defined without even having a geometric model. This can be done by merely giving the set of elements and the result of combining any two of them. Each of the following three tables defines an algebraic system.

(a)

H	R	W
R	R	W
W	R	W

(b)

*	P_0	P_1	P_2	P_3
P_0	P_2	P_3	P_0	P_1
P_1	P_3	P_2	P_1	P_0
P_2	P_0	P_1	P_2	P_3
P_3	P_1	P_0	P_3	P_2

(c)

~	△	▭	X	V
△	△	▭	X	V
▭	▭	X	V	△
X	X	V	△	▭
V	V	△	▭	X

Exercises - 6

1. Use these tables to complete the following statements correctly.

   (a) $W \pi R =$

   (b) $\triangle \sim X =$

   (c) $\vee \sim \vee =$

   (d) $R \pi W =$

   (e) $P_1 * P_2 =$

   (f) $P_2 * P_3 =$

   (g) $P_0 * P_2 =$

   (h) $\square \sim X =$

   (i) $\vee \sim \square =$

   (j) $W \pi W =$

   (k) $R \pi R =$

   (l) $P_3 * P_3 =$

2. Which one, or ones, of the binary operations π, $*$, $\sim$ has an identity element? What is it in each case?

3. Which one, or ones, of the operations π, $*$, $\sim$ is commutative? Prove your statement.

4. Use the tables to "compute" the following [Assume that parentheses, (), mean that the quantity enclosed by them is to be computed first and then treated as a single element]:

   (a) $P_0 * (P_1 * P_2) =$ ____

   (b) $(P_0 * P_1) * P_2 =$ ____

   (c) $P_0 * (P_1 * P_3) =$ ____

   (d) $(P_0 * P_1) * P_3 =$ ____

   (e) $(P_2 * P_0) * P_3 =$ ____

   (f) $P_2 * (P_0 * P_3) =$ ____

   (g) $\triangle \sim (\triangle \sim X) =$ ____

   (h) $(\triangle \sim \triangle) \sim X =$ ____

   (i) $(\vee \sim \square) \sim \triangle =$ ____

   (j) $\vee \sim (\square \sim \triangle) =$ ____

   Does either table (b) or table (c) seem to represent an associative operation? Why? How could you prove your statement? What would another person have to do to prove you wrong?

Systems of Natural Numbers and Whole Numbers

The natural numbers form an algebraic system under both the operations of addition and multiplication. What are some of their properties? The natural numbers have an identity element with respect to multiplication. What is it? Do they have an identity element with respect to addition?

If we take the system of natural numbers and one more element, the number zero, we get a new and different mathematical system called the system of whole numbers. Which of the properties listed for the natural numbers are also possessed by the whole numbers? Do the whole numbers have any additional properties?

Exercises - 7

1. List the properties of these mathematical systems. How are they the same? In what ways are they different?
   (a) The system of natural numbers and addition and multiplication.
   (b) The system of whole numbers under addition and multiplication.
   (c) The system whose set is the set of odd numbers and whose operation is multiplication.
   (d) The system whose set is the set made up of zero and the multiples of 3, and whose operation is multiplication.
   (e) The system whose set is the set made up of zero and the multiples of 3, and whose operation is addition.
   (f) The system whose set is the even numbers and whose operation is addition.
   (g) The system whose set is the fractions between 0 and 1 and whose operation is multiplication.
   (h) The same set as in (g) under the operation of addition.
2. Make up an algebraic system (a combination of a set and an operation) of your own. Make at least a partial table for your system. (Could you make complete tables for the operations in Exercise 7 as we could for those in Exercise 6? Why?) List the properties of your system.

More about Modular Arithmetic

We have seen modular arithmetic for addition. If we put in multiplication we will get a different mathematical system. With both operations, modular arithmetic will be more like ordinary arithmetic.

Complete the multiplication tables for

mod 5 and mod 8

X	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	-	-
4	0	4	-	-	-

X	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	2	4	6	0	2	4	6
3	0	3	6	1	-	-	-	-
4	0	4	-	-	-	-	-	-
5	0	5	-	-	-	-	-	-
6	0	6	-	-	-	-	-	-
7	0	-	-	-	-	-	-	-

List the properties. (commutative, associative, closure, identity, inverse). Does the distributive property for multiplication over addition hold? Is it true that if a product is zero at least one of the factors is zero in mod 5? in mod 8?

Modular arithmetics may be thought of as mathematical systems with two operations. Just as we can solve problems using ordinary arithmetic, we can solve problems using modular arithmetic.

Exercises - 8

1. Find the sum of:

A	B
(a) 1 and 5 mod 8	(a) 3 and 2 mod 8
(b) 4 and 3 mod 8	(b) 6 and 5 mod 8
(c) 4 and 4 mod 8	(c) 7 and 7 mod 8
(d) 4 and 5 mod 8	(d) 0 and 2 mod 8
(e) 0 and 6 mod 8	(e) 7, 3, 5 and 1 mod 8
(f) 6 and 7 mod 8	(f) 6, 7, 7 and 5 mod 8

(g) 3, 5 and 2 mod 8

(h) 7, 6 and 4 mod 8

(i) 3, 7 and 6 mod 8

(j) 4 and 2 mod 5

(k) 7 and 2 mod 8

(l) 7 and 2 mod 9

(m) 7 and 2 mod 10

(n) the first three even numbers, mod 9 (0, 2 and 4)

(o) the multiples of three that are between 5 and 10 in arithmetic mod 12

(g) 5 and 4 mod 7

(h) 6, 3, 5 and 5 mod 7

(i) 10, 8, 6 mod 12

(j) 12, 3, 9, 2 mod 15

(k) the first four even numbers mod 12 (0, 2, 4 and 6)

(l) the first three prime numbers mod 9

(m) the multiples of three that are between 0 and 14 in arithmetic mod 15

(n) the greatest common factor of 4, 6 and 8, and the greatest common factor of 6 and 9 in arithmetic mod 12

(o) the numbers less than 10 in arithmetic mod 13.

2. Find the products:

(a) 3 x 5 mod 8

(b) 2 x 3 mod 8

(c) 2 x 3 mod 4

(d) 2 x 3 mod 5

(e) 2 x 3 mod 6

(f) 5 x 8 mod 7

(g) 3^2 mod 5

(h) 7^2 mod 8

(i) 6 x 4 mod 5

(j) 3 x 4 x 6 mod 9

(k) 4^3 mod 5

(a) 2 x 7 mod 8

(b) 5 x 3 mod 8

(c) 5 x 3 mod 9

(d) 5 x 3 mod 10

(e) 12 x 14 mod 18

(f) 6^2 mod 8

(g) 10^2 mod 12

(h) 7 x 6 x 7 mod 9

(i) 8 x 2 x (5 + 4) mod 11

(j) (3 + 4) x (9 - 5) x (5 - 2) mod 12

(k) 5^3 mod 9

3. Find the quotients:

Remember that division is always defined after we know about multiplication. Thus, in ordinary arithmetic the question "six divided by 2 is what?" means, really, "six is obtained by multiplying 2 by what?" An operation which begins with one of the numbers and the "answer" to another binary operation and asks for the other number is called an inverse operation. Division is the inverse operation of multiplication.

(a) $\frac{2}{3}$ mod 8
(b) $\frac{7}{6}$ mod 8
(c) $\frac{6}{2}$ mod 8
(d) $\frac{3}{4}$ mod 5
(e) $\frac{0}{2}$ mod 5
(f) $\frac{0}{2}$ mod 8

(a) $\frac{6}{5}$ mod 8
(b) $\frac{6}{2}$ mod 8
(c) $\frac{0}{4}$ mod 8
(d) $\frac{0}{4}$ mod 5
(e) $\frac{3}{2}$ mod 5
(f) $\frac{7}{3}$ mod 10

4. Compute: Remember, subtraction is the inverse operation to addition.

(a) 7 - 3 mod 8
(b) 3 - 7 mod 8
(c) 9 - 2 mod 10
(d) 2 - 9 mod 10
(e) 3 - 4 mod 5
(f) 3 - 4 mod 8
(g) 2 - 5 mod 9
(h) 3 - 6 mod 9
(i) 4 - 7 mod 9
(j) 4 - 8 mod 9
(k) Does (4-7)=(1-4) in arithmetic mod 8?
(l) Does (6-2)=(3-9) in arithmetic mod 10?

(a) 7 - 5 mod 8
(b) 5 - 7 mod 8
(c) 10 - 3 mod 11
(d) 3 - 10 mod 11
(e) 2 - 5 mod 6
(f) 2 - 5 mod 10
(g) 2 - 6 mod 12
(h) 3 - 7 mod 12
(i) 4 - 8 mod 12
(j) 4 - 9 mod 12
(k) For what modulus does 1 - 3 = 5?
(l) For what modulus does 5 - 9 = 10?

5. Study this modular arithmetic table:

x	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

(a) What is 5 x 5 mod 6? In this product the factors were the same. When a product has two identical factors we call one of them the square root of the product. 5 is the square root of 1 in arithmetic mod 6. Are any other examples like this listed in the table?

(b) Does 1 have any square roots other than 5?

(c) Does every number have two different square roots?

(d) Does any number have just one square root?

(e) Does any number have no square roots at all?

(f) Fill in this chart:

Number	Square Roots of the Number
0	
1	
2	
3	
4	
5	

6. Consider the system of natural numbers.

(a) Can you find a number that has a square root? What is it?

(b) Does more than one number have a square root?

(c) Does every number have a square root? Prove it.

(d) Does any natural number have more than one square root?

(e) Fill in the chart with names of natural numbers less than 110.

Number	1	4	9
Square Roots of the Number	1	2	3